Appendix A1: Proof of the Characterization of Optimal Tasks in Section 2.2

Let $\hat{\tau}(\tilde{m}')$ denote the cutoff talent below which the second period income from trying the task $\tilde{m}'$ is lower than the outside option $\bar{y} = 1$: $\hat{\tau}(\tilde{m}') \cdot \tilde{\gamma}(\tilde{m}')^{\tilde{\alpha}} - \tilde{\alpha}(\tilde{m}')^{\tilde{\gamma}} = 1$. Some properties of $\hat{\tau}(\tilde{m}')$ are $\hat{\tau}(1) = 1$; $\hat{\tau}(\tilde{m}')$ rises as $\tilde{m}'$ moves away from 1; and $1 < \hat{\tau}(\tilde{m}') < \tilde{m}'$ if $\tilde{m} > 1$. This implies that if the updated talent for the initial task $\tilde{\tau} \leq 1$, the worker moves to another trade, assuming the tie-breaking rule that the worker moves when staying and moving deliver the same future income. If $\tilde{\tau} > 1$, there are $\hat{m}_l(\tilde{\tau}) < 1$ and $\hat{m}_h(\tilde{\tau}) > \tilde{\tau}$ such that $\tilde{\tau} > \hat{\tau}(\tilde{m}')$ iff $\hat{m}_l(\tilde{\tau}) < \tilde{m}' < \hat{m}_h(\tilde{\tau})$. Then, the worker stays in the initial trade iff the set $S \equiv \{\tilde{m}' | \; \tilde{m}' \in (\hat{m}_l(\tilde{\tau}), \hat{m}_h(\tilde{\tau})) \cap [\eta_l \tilde{m}, \eta_h \tilde{m}]\}$ is not empty. Within this set, the second-period income y' rises in $\tilde{m}'$ if $\tilde{m}' < \tilde{\tau}$, and falls in $\tilde{m}'$ if $\tilde{m}' > \tilde{\tau}$. Therefore, if $\tilde{\tau} > 1$ and S is not empty, the second-period income is maximized by choosing $\tilde{m}' = \tilde{\tau}$ if $\tilde{\tau} \in [\eta_l \tilde{m}, \eta_h \tilde{m}]$; $\tilde{m}' = \eta_l \tilde{m}$ if $\tilde{\tau} < \eta_l \tilde{m}$; and $\tilde{m}' = \eta_h \tilde{m}$ if $\tilde{\tau} > \eta_h \tilde{m}$.

If the initial task $\tilde{m} < 1$, the worker can raise the first-period income y by raising $\tilde{m}$ without lowering the second-period income y' conditional on any $\tilde{\tau}$. If $\tilde{m} = 1$, the worker can raise the second-period income y' conditional on $\tilde{\tau} > \eta_h \tilde{m}$ by raising $\tilde{m}$ without lowering the first-period income y or the second-period income y' conditional on $\tilde{\tau} \leq \eta_h \tilde{m}$. Therefore, we can set $\tilde{m} > 1$ without a loss of generality. Then, the set S is not empty iff $\eta_l \tilde{m} < \hat{m}_h(\tilde{\tau})$, which defines the talent threshold $\check{\tau}(\tilde{m})$ for the first-period task $\tilde{m}$ above which the worker stays in the initial trade. If $\eta_l \tilde{m} \leq 1$, $\check{\tau}(\tilde{m}) = 1$. If $\eta_l \tilde{m} > 1$, $\check{\tau}(\tilde{m}) \in (1, \eta_l \tilde{m})$ solves $\eta_l \tilde{m} = \hat{m}_h(\check{\tau}(\tilde{m}))$, that is, $\check{\tau}(\tilde{m}) \cdot \tilde{\gamma}(\eta_l \tilde{m})^{\tilde{\alpha}} - \tilde{\alpha}(\eta_l \tilde{m})^{\tilde{\gamma}} = 1$ or (1) in the main text.

Appendix A2: Proof of a Unique Optimal $(\tilde{m}, \check{\tau})$ under the Pareto Distribution of Talent in Section 2.2

Substituting $F(\tilde{\tau}) = 1 - k\tilde{\tau}^{-n}/n$ in (2) and rearranging terms using $\bar{\tau} = n/(n-1) \cdot (k/n)^{1/n}$ $= 1$, we obtain

$$\check{\tau}^n - \beta\eta_l^{\tilde{\gamma}}\left(\frac{k}{n-1} - \frac{k}{n}\right)\left(\frac{\check{\tau}}{\eta_l\tilde{m}}\right)^n\left(\frac{\eta_l}{\eta_h}\right)^{n-\tilde{\gamma}} - \frac{\check{\tau}}{\eta_l\tilde{m}}\cdot\eta_l\check{\tau}^{n-1} < 0. \quad \text{(A1)}$$

Similarly, from (3), we obtain

$$\check{\tau}^n + \beta\eta_l^{\tilde{\gamma}}\frac{k}{n} + \beta\eta_l^{\tilde{\gamma}}\left(\frac{k}{n-1} - \frac{k}{n}\right)\left(\frac{\check{\tau}}{\eta_l\tilde{m}}\right)^n\left(1 - \left(\frac{\eta_l}{\eta_h}\right)^{n-\tilde{\gamma}}\right) - \frac{\check{\tau}}{\eta_l\tilde{m}}\left(\eta_l\check{\tau}^{n-1} + \beta\eta_l^{\tilde{\gamma}}\frac{k}{n-1}\right) < 0. \quad \text{(A2)}$$

Let LH and $\tilde{LH}$ denote the lefthand sides of (A1) and (A2), respectively. We can think of LH and $\tilde{LH}$ as functions of two variables, $\check{\tau}$ and $\check{\tau}/(\eta_l\tilde{m})$. We can show that in (A2), holding $\check{\tau}$, $\tilde{LH}$ is positive when $\check{\tau}/(\eta_l\tilde{m}) = 0$, and declines as $\check{\tau}/(\eta_l\tilde{m})$ rises while $\check{\tau}/(\eta_l\tilde{m}) \leq 1$. In (A1), $LH = \tilde{LH}$ when $\check{\tau}/(\eta_l\tilde{m}) = 1$, declines as $\check{\tau}/(\eta_l\tilde{m})$ rises, and becomes negative before $\check{\tau}/(\eta_l\tilde{m}) = \check{\tau}/\eta_l$. Therefore, given $\check{\tau}$, there is a unique value of $\eta_l\tilde{m}/\check{\tau}$ that solves (A1) or (A2) with equality, as expected. In (A2), holding $\check{\tau}/(\eta_l\tilde{m})$, $\tilde{LH}$ rises in $\check{\tau}$ if $1 < \check{\tau} < \eta_l\tilde{m}$. This implies that if (4) does not hold, $\check{\tau}/(\eta_l\tilde{m})$ that solves (A2) with equality rises in $\check{\tau}$ or equivalently, $\tilde{m}/\check{\tau}$ falls in $\check{\tau}$. This proves that if talent is distributed by a Pareto distribution and if (4) does not hold, there is a unique crossing point $(\tilde{m}, \check{\tau})$ that solves $\tilde{m} = \tilde{m}(\check{\tau})$ and $\check{\tau} = \check{\tau}(\tilde{m})$.

Appendix A3: The Effects of a Rising $\tilde{\gamma}$ on y' in Section 3

As $\tilde{\gamma}$ rises, y' stays at one if $\tilde{\tau} \leq \check{\tau}$ (the first-segment), and $y' = \tilde{\tau}^{\tilde{\gamma}}$ rises if $\tilde{\tau} \in [\eta_l\tilde{m}, \eta_h\tilde{m}]$ (the third segment). If $\tilde{\tau} > \eta_h\tilde{m}$ (the fourth segment), a rising $\tilde{\gamma}$ raises $y' = \tilde{\tau}\tilde{\gamma}(\eta_h\tilde{m})^{\tilde{\alpha}} - \tilde{\alpha}(\eta_h\tilde{m})^{\tilde{\gamma}}$ by raising it holding $\eta_h\tilde{m}$ and by raising $\eta_h\tilde{m}$. If $\tilde{\tau} \in (\check{\tau}, \eta_l\tilde{m})$ (the second segment), $y' = \tilde{\tau}\tilde{\gamma}(\eta_l\tilde{m})^{\tilde{\alpha}} - \tilde{\alpha}(\eta_l\tilde{m})^{\tilde{\gamma}}$ and $\partial y'/\partial\tilde{\gamma} \geq 0$ if and only if $\tilde{\tau} \geq z_0 \equiv \eta_l\tilde{m}\cdot(1 + \tilde{\alpha}\log(\eta_l\tilde{m}))/(1 + \tilde{\gamma}\log(\eta_l\tilde{m}))$ where $z_0 \in (\check{\tau}, \eta_l\tilde{m})$. Further, $\partial y'/\partial\tilde{\gamma}$ rises in $\tilde{\tau}$; $\partial y'/\partial(\eta_l\tilde{m})$ rises in $\tilde{\tau}$ and is equal to zero when $\tilde{\tau} = \eta_l\tilde{m}$. Then, $dy'/d\tilde{\gamma} = \partial y'/\partial\tilde{\gamma} + (d(\eta_l\tilde{m})/d\tilde{\gamma})\cdot(\partial y'/\partial(\eta_l\tilde{m})) = 0$ for some $z_1 \in (z_0, \eta_l\tilde{m})$, and $dy'/d\tilde{\gamma}$ rises in $\tilde{\tau}$. Therefore, in the second segment, y' falls if $\tilde{\tau} \in (\check{\tau}, z_1)$, and rises if $\tilde{\tau} \in (z_1, \eta_l\tilde{m})$. Figure 4 visualizes the changes of y'.

Appendix A4: Proof that $d\tilde{m}/d\tilde{\gamma} < 0$ and $d\check{\tau}/d\tilde{\gamma} < 0$ if $\check{\tau} > 1$ in Proposition 2

Given $y = \tilde{\gamma}\tilde{m}^{\tilde{\alpha}} - \tilde{\alpha}\tilde{m}^{\tilde{\gamma}} = \omega$, we have $dy/d\tilde{\gamma} = \partial y/\partial\tilde{\gamma} + (d\tilde{m}/d\tilde{\gamma}) \cdot (\partial y/\partial\tilde{m}) = 0$, from which we can derive

$$\frac{d\tilde{m}}{d\tilde{\gamma}} = -\frac{\tilde{m}}{\tilde{\alpha}\tilde{\gamma}} - \frac{\tilde{m}}{\tilde{\gamma}}\log\tilde{m} + \frac{1}{\tilde{\alpha}\tilde{\gamma}} \cdot \frac{\tilde{m}\log\tilde{m}}{\tilde{m}-1} < 0. \tag{A3}$$

If $\check{\tau} > 1$ or equivalently $\eta_l\tilde{m} > 1$, using (1), we can derive

$$\frac{d\check{\tau}}{d\tilde{\gamma}} = \frac{\partial\check{\tau}}{\partial\tilde{\gamma}} + \frac{d\tilde{m}}{d\tilde{\gamma}} \cdot \frac{\partial\check{\tau}}{\partial\tilde{m}} = \frac{1}{\tilde{\gamma}^2} \cdot \eta_l\tilde{m} \cdot \Omega \tag{A4}$$

where

$$\begin{aligned}\Omega &\equiv (\eta_l\tilde{m})^{-\tilde{\gamma}}\log(\eta_l\tilde{m})^{-\tilde{\gamma}} + \left(\frac{\log\tilde{m}}{\tilde{m}-1} - \tilde{\alpha}\log\tilde{m}\right)(1-(\eta_l\tilde{m})^{-\tilde{\gamma}}) \\ &\leq (\eta_l\tilde{m})^{-\tilde{\gamma}}\log(\eta_l\tilde{m})^{-\tilde{\gamma}} + \left(\frac{\log\eta_l\tilde{m}}{\eta_l\tilde{m}-1} - \tilde{\alpha}\log\eta_l\tilde{m}\right)(1-(\eta_l\tilde{m})^{-\tilde{\gamma}}) \\ &< 0.\end{aligned} \tag{A5}$$

Appendix A5: Proof that $dy'/d\tilde{\gamma} \geq 0$ at any $\tilde{\tau}$ in Proposition 2

As $\tilde{\gamma}$ rises, y' stays at one if $\tilde{\tau} < \check{\tau}$ (the first-segment), and $y' = \tilde{\tau}^{\tilde{\gamma}}$ rises if $\tilde{\tau} \in [\eta_l\tilde{m}, \eta_h\tilde{m}]$ (the third segment). If $\tilde{\tau} \in [\check{\tau}, \eta_l\tilde{m})$ (the second segment), $y' = \tilde{\tau}\tilde{\gamma}(\eta_l\tilde{m})^{\tilde{\alpha}} - \tilde{\alpha}(\eta_l\tilde{m})^{\tilde{\gamma}}$ rises both when $\tilde{\tau} = \eta_l\tilde{m}$ and when $\tilde{\tau} = \check{\tau}$ given the falling $\check{\tau}$, which implies that y' rises for all $\tilde{\tau}$ between them. If $\tilde{\tau} > \eta_h\tilde{m}$ (the fourth segment), $y' = \tilde{\tau}\tilde{\gamma}(\eta_h\tilde{m})^{\tilde{\alpha}} - \tilde{\alpha}(\eta_h\tilde{m})^{\tilde{\gamma}}$ and rises when $\tilde{\tau} = \eta_h\tilde{m}$. Given $y = \tilde{\gamma}\tilde{m}^{\tilde{\alpha}} - \tilde{\alpha}\tilde{m}^{\tilde{\gamma}} = \omega$, we can show

$$\frac{dy}{d\tilde{\gamma}} = \frac{d(\tilde{\gamma}\tilde{m}^{\tilde{\alpha}})}{d\tilde{\gamma}} - \frac{d(\tilde{\alpha}\tilde{m}^{\tilde{\gamma}})}{d\tilde{\gamma}} = -(\tilde{m}-1)\frac{d(\tilde{\gamma}\tilde{m}^{\tilde{\alpha}})}{d\tilde{\gamma}} + m^{\tilde{\gamma}}\log\tilde{m} = 0, \tag{A6}$$

which implies that $d(\tilde{\gamma}\tilde{m}^{\tilde{\alpha}})/d\tilde{\gamma} > 0$. Then, $y'|_{\tilde{\tau}>\eta_h\tilde{m}} - y'|_{\tilde{\tau}=\eta_h\tilde{m}}$ rises as $\tilde{\gamma}$ rises. Therefore, y' rises for all $\tilde{\tau} > \eta_h\tilde{m}$.

Appendix A6: The Effects of the Changes of η_h and η_l in Section 4.1

First, consider how the optimal $(\tilde{m}, \check{\tau})$ changes as η_h changes. In (2) and (3), the weighted undermatch factor $\beta\eta_h^{\tilde{\alpha}}\int_{\eta_h\tilde{m}}^{\infty}(\tilde{\tau}-\eta_h\tilde{m})dF(\tilde{\tau})$ rises as η_h falls iff

$$\frac{E[\tilde{\tau}|\tilde{\tau}\geq\eta_h\tilde{m}]}{\eta_h\tilde{m}}<\frac{\tilde{\gamma}}{\tilde{\gamma}-1}. \tag{A7}$$

As discussed in section 2.2, the undermatch factor $\int_{\eta_h\tilde{m}}^{\infty}(\tilde{\tau}-\eta_h\tilde{m})dF(\tilde{\tau})$ falls in η_h holding $\tilde{m}$, while the undermatch factor weight $\eta_h^{\tilde{\alpha}}$ rises in η_h. The first effect dominates the second effect iff $\tilde{\gamma}=1+\tilde{\alpha}$ is small enough. Assuming the Pareto distribution of talent, $E[\tilde{\tau}|\tilde{\tau}\geq\eta_h\tilde{m}]/\eta_h\tilde{m}=n/(n-1)<\tilde{\gamma}/(\tilde{\gamma}-1)$ iff $\tilde{\gamma}<n$. In figure 3, the reaction function $\tilde{m}(\check{\tau})$ shifts up if $\tilde{\gamma}<n$, and shifts down if $\tilde{\gamma}>n$, while the reaction function $\check{\tau}(\tilde{m})$ stays constant. Therefore, as η_h falls, the optimal $\tilde{m}$ rises if $\tilde{\gamma}<n$, and falls if $\tilde{\gamma}>n$. The optimal $\check{\tau}$ moves in the same direction in a type 2 solution.

We can also show that $\eta_h\tilde{m}$ falls as η_h falls. This is easy to see in (2) for a type 1 solution. For a type 2 solution, assuming the Pareto distribution of talent, Appendix A6.1 shows that $\eta_h\tilde{m}$ falls as η_h falls even if $\tilde{\gamma}<n$: The direct effect of a falling η_h dominates the indirect effect of a rising $\tilde{m}$. In summary, we have the following proposition.

Proposition A1. Assume the Pareto distribution of talent. As η_h falls, the optimal $\tilde{m}$ rises if $\tilde{\gamma}<n$, and falls if $\tilde{\gamma}>n$. As η_h falls while $\check{\tau}>1$ (type 2 solution), the optimal $\check{\tau}$ rises if $\tilde{\gamma}<n$, and falls if $\tilde{\gamma}>n$. As η_h falls, the optimal $\eta_h\tilde{m}$ falls.

Now, consider how the optimal $(\tilde{m},\check{\tau})$ changes as η_l rises. In (3), both the overmatch factor $\int_{\check{\tau}}^{\eta_l\tilde{m}}(\eta_l\tilde{m}-\tilde{\tau})dF(\tilde{\tau})$ and the overmatch factor weight $\eta_l^{\tilde{\alpha}}$ rise in η_l. In (1), holding $\tilde{m}$, as η_l rises holding $\tilde{m}$, $\check{\tau}$ rises. In figure 3, the optimal $\tilde{m}$ and $\check{\tau}$ do not change in a type 1 solution while $\tilde{m}(\check{\tau})$ shifts down while $\check{\tau}(\tilde{m})$ shifts to the right in a type 2 solution, leaving the possibility of a rising $\tilde{m}$ or a falling $\check{\tau}$. Assuming the Pareto distribution of talent, we can show that the overall effect is a falling $\tilde{m}$ and a rising $\check{\tau}$. We can also show that $\eta_l\tilde{m}$ rises as η_l rises: the direct effect of rising η_l dominates any indirect effects of falling $\tilde{m}$. See Appendices A6.2 and A6.3. In summary, we have the following proposition.

Proposition A2. As η_l rises, the optimal $\tilde{m}$ and $\check{\tau}$ stay constant in a type 1 solution. Assume the Pareto distribution of talent for the following statements. As η_l rises, the optimal $\tilde{m}$ falls and the optimal $\check{\tau}$ rises in a type 2 solution. As η_l rises, the optimal $\eta_l\tilde{m}$ rises.

Proposition 3 follows from Propositions A1 and A2.

Appendix A6.1: Proof that $d(\eta_h\tilde{m})/d\eta_h > 0$ in a Type 2 Solution in Proposition A1

Rewrite (A2) with equality as follows.

$$\begin{aligned}\Phi(\eta_h,\check{\tau},\tilde{m}) \equiv\ & 1-\frac{1}{\tilde{m}}+\beta\eta_l^{\tilde{\gamma}}\left(\frac{k}{n-1}-\frac{k}{n}\right)\left(\frac{1}{\eta_l\tilde{m}}\right)^n\left(1-\left(\frac{\eta_l}{\eta_h}\right)^{n-\tilde{\gamma}}\right)\\ & -\beta\eta_l^{\tilde{\gamma}}\frac{1}{\check{\tau}^n}\left(\frac{k}{n-1}\cdot\frac{\check{\tau}}{\eta_l\tilde{m}}-\frac{k}{n}\right)\\ =\ & 0.\end{aligned} \tag{A8}$$

We have

$$\frac{\partial\Phi}{\partial\eta_h}=\beta\eta_l^{\tilde{\gamma}}k\cdot\frac{n-\tilde{\gamma}}{n(n-1)}\left(\frac{1}{\eta_l\tilde{m}}\right)^n\left(\frac{\eta_l}{\eta_h}\right)^{n-\tilde{\gamma}}\frac{1}{\eta_h}; \tag{A9}$$

$$\frac{\partial\Phi}{\partial\check{\tau}}=-\beta\eta_l^{\tilde{\gamma}}k\cdot\frac{1}{\check{\tau}^{n+1}}\left(1-\frac{\check{\tau}}{\eta_l\tilde{m}}\right); \tag{A10}$$

and

$$\frac{\partial\Phi}{\partial\tilde{m}}=\frac{1}{\tilde{m}^2}+\beta\eta_l^{\tilde{\gamma}}k\cdot\frac{1}{n-1}\cdot\frac{1}{\check{\tau}^n}\cdot\frac{\check{\tau}}{\eta_l\tilde{m}}\cdot\frac{1}{\tilde{m}}\left(1-\left(\frac{\check{\tau}}{\eta_l\tilde{m}}\right)^{n-1}\left(1-\left(\frac{\eta_l}{\eta_h}\right)^{n-\tilde{\gamma}}\right)\right). \tag{A11}$$

Dividing both sides of (1) by $\eta_l\tilde{m}$ and using $\tilde{\alpha}=\tilde{\gamma}-1$, we have

$$\tilde{\gamma}\left(1-\frac{\check{\tau}(\tilde{m})}{\eta_l\tilde{m}}\right)=1-\left(\frac{1}{\eta_l\tilde{m}}\right)^{\tilde{\gamma}}. \tag{A12}$$

Using (A12), we have

$$d\check{\tau}=\frac{\tilde{\alpha}}{\tilde{\gamma}}\left(1-\left(\frac{1}{\eta_l\tilde{m}}\right)^{\tilde{\gamma}}\right)d(\eta_l\tilde{m})=(\tilde{\gamma}-1)\left(1-\frac{\check{\tau}}{\eta_l\tilde{m}}\right)\eta_l d\tilde{m}. \tag{A13}$$

Combining (A8), (A9), (A10), (A11), and (A13), we have

$$d\Phi = d\check{\tau}\frac{\partial \Phi}{\partial \check{\tau}} + d\eta_h \frac{\partial \Phi}{\partial \eta_h} + d\tilde{m}\frac{\partial \Phi}{\partial \tilde{m}} = d\eta_h \frac{\partial \Phi}{\partial \eta_h} + d\tilde{m}\left(\frac{\beta \eta_l^{\tilde{\gamma}} k}{\check{\tau}^n \tilde{m}}\Psi_{\tilde{m}} + \frac{1}{\tilde{m}^2}\right) = 0 \qquad \text{(A14)}$$

where

$$\Psi_{\tilde{m}} = -(\tilde{\gamma}-1)\left(1-\frac{\check{\tau}}{\eta_l \tilde{m}}\right)^2 \frac{\eta_l \tilde{m}}{\check{\tau}} + \frac{1}{n-1}\cdot\frac{\check{\tau}}{\eta_l \tilde{m}}\left(1-\left(\frac{\check{\tau}}{\eta_l \tilde{m}}\right)^{n-1}\left(1-\left(\frac{\eta_l}{\eta_h}\right)^{n-\tilde{\gamma}}\right)\right). \qquad \text{(A15)}$$

Now, we have $\check{\tau}/\eta_l \tilde{m} > (\tilde{\gamma}-1)/\tilde{\gamma}$ from (A12). In (A2) with equality, noting the two terms without the factor $\beta \eta_l^{\tilde{\gamma}}$ on the lefthand side sum to a positive value, we have

$$\frac{\check{\tau}}{\eta_l \tilde{m}} > \frac{n-1}{n} + \frac{1}{n}\left(\frac{\check{\tau}}{\eta_l \tilde{m}}\right)^n \left(1-\left(\frac{\eta_l}{\eta_h}\right)^{n-\tilde{\gamma}}\right). \qquad \text{(A16)}$$

Then, if $\tilde{\gamma} < n$, $\check{\tau}/\eta_l \tilde{m} > (n-1)/n > (\tilde{\gamma}-1)/\tilde{\gamma}$. If $\tilde{\gamma} > n$, $\check{\tau}/\eta_l \tilde{m} > (\tilde{\gamma}-1)/\tilde{\gamma} > (n-1)/n$.

Using (A15) and (A16), we can show

$$\Psi_{\tilde{m}} > -(\tilde{\gamma}-1)\left(\frac{\eta_l \tilde{m}}{\check{\tau}} - 1\right) + \tilde{\gamma}\left(1 - \frac{\check{\tau}}{\eta_l \tilde{m}}\right) > 0 \qquad \text{(A17)}$$

for all $\check{\tau}/(\eta_l \tilde{m}) \in ((\tilde{\gamma}-1)/\tilde{\gamma}, 1)$. Then, from (A14), $d(\eta_h \tilde{m})/d\eta_h = \tilde{m} + \eta_h d\tilde{m}/d\eta_h > 0$ iff

$$\frac{d\tilde{m}}{d\eta_h} = -\frac{\partial \Phi/\partial \eta_h}{\beta \eta_l^{\tilde{\gamma}} k/(\check{\tau}^n \tilde{m})\cdot \Psi_{\tilde{m}} + 1/\tilde{m}^2} > -\frac{\tilde{m}}{\eta_h}. \qquad \text{(A18)}$$

The inequality in (A18) holds if

$$\Lambda \equiv \Psi_{\tilde{m}} - \frac{n-\tilde{\gamma}}{n(n-1)}\left(\frac{\check{\tau}}{\eta_l \tilde{m}}\right)^n \left(\frac{\eta_l}{\eta_h}\right)^{n-\tilde{\gamma}} > 0. \qquad \text{(A19)}$$

We have $\Lambda > 0$ if $\tilde{\gamma} \geq n$. If $\tilde{\gamma} < n$, consulting (A15) and (A16), we have

$$\begin{aligned}\Lambda =& -(\tilde{\gamma}-1)\left(1-\frac{\check{\tau}}{\eta_l \tilde{m}}\right)^2 \frac{\eta_l \tilde{m}}{\check{\tau}} + \frac{1}{n-1}\cdot\frac{\check{\tau}}{\eta_l \tilde{m}}\left(1-\left(\frac{\check{\tau}}{\eta_l \tilde{m}}\right)^{n-1}\left(1-\frac{\tilde{\gamma}}{n}\left(\frac{\eta_l}{\eta_h}\right)^{n-\tilde{\gamma}}\right)\right)\\ >& -(\tilde{\gamma}-1)\left(1-\frac{\check{\tau}}{\eta_l \tilde{m}}\right)^2 \frac{\eta_l \tilde{m}}{\check{\tau}} + \frac{1}{n-1}\cdot\frac{\check{\tau}}{\eta_l \tilde{m}} - \frac{1}{n-1}\left(\frac{\check{\tau}}{\eta_l \tilde{m}}\right)^n\\ &+ \frac{\tilde{\gamma}}{n}\left(1+\frac{1}{n-1}\left(\frac{\check{\tau}}{\eta_l \tilde{m}}\right)^n - \frac{n}{n-1}\cdot\frac{\check{\tau}}{\eta_l \tilde{m}}\right)\\ \equiv& \ \tilde{\Lambda}.\end{aligned} \qquad \text{(A20)}$$

Consider $\tilde{\Lambda}$ as a function of $\check{\tau}/(\eta_l \tilde{m})$. We can show that $\tilde{\Lambda} > 0$ when $\check{\tau}/(\eta_l \tilde{m}) = (n-1)/n$; $\tilde{\Lambda} = 0$ when $\check{\tau}/(\eta_l \tilde{m}) = 1$; and $\partial\tilde{\Lambda}/\partial(\check{\tau}/\eta_l \tilde{m})$ falls as $\check{\tau}/(\eta_l \tilde{m})$ rises. These properties of $\tilde{\Lambda}$ imply that $\tilde{\Lambda} > 0$ for all $\check{\tau}/(\eta_l \tilde{m}) \in ((n-1)/n, 1)$, which in turn implies that $d(\eta_h \tilde{m})/d\eta_h > 0$.

Appendix A6.2: Proof that $d\tilde{m}/d\eta_l < 0$ in a Type 2 solution in Proposition A2

Consider Φ in (A8) as a function of η_l, $\check{\tau}$, and $\tilde{m}$: $\Phi = \Phi(\eta_l, \check{\tau}, \tilde{m}) = 0$. Then, (A10) and (A11) are valid;

$$\frac{\partial \Phi}{\partial \eta_l} = -\beta \eta_l^{\tilde{\alpha}} k(n-\tilde{\gamma})\left(\frac{1}{n-1} - \frac{1}{n}\right)\left(\frac{1}{\eta_l \tilde{m}}\right)^n + \beta \eta_l^{\tilde{\alpha}} k \cdot \frac{1}{\check{\tau}^n}\left(\frac{\tilde{\gamma}}{n} - \frac{\tilde{\gamma}-1}{n-1} \cdot \frac{\check{\tau}}{\eta_l \tilde{m}}\right); \quad \text{(A21)}$$

and (A13) becomes

$$d\check{\tau} = \frac{\tilde{\alpha}}{\tilde{\gamma}}\left(1 - \left(\frac{1}{\eta_l \tilde{m}}\right)^{\tilde{\gamma}}\right) d(\eta_l \tilde{m}) = (\tilde{\gamma}-1)\left(1 - \frac{\check{\tau}}{\eta_l \tilde{m}}\right)(\tilde{m} d\eta_l + \eta_l d\tilde{m}). \quad \text{(A22)}$$

Combining (A10), (A11), (A21), (A22), and $\Phi(\eta_l, \check{\tau}, \tilde{m}) = 0$, we have

$$d\Phi = d\check{\tau} \cdot \frac{\partial \Phi}{\partial \check{\tau}} + d\eta_l \cdot \frac{\partial \Phi}{\partial \eta_l} + d\tilde{m} \cdot \frac{\partial \Phi}{\partial \tilde{m}} = d\eta_l \cdot \frac{\beta \eta_l^{\tilde{\alpha}} k}{\check{\tau}^n} \cdot \Psi_{\eta_l} + d\tilde{m}\left(\frac{\beta \eta_l^{\tilde{\gamma}} k}{\check{\tau}^n \tilde{m}} \cdot \Psi_{\tilde{m}} + \frac{1}{\tilde{m}^2}\right) = 0 \quad \text{(A23)}$$

where

$$\begin{aligned}\Psi_{\eta_l} &= -(\tilde{\gamma}-1)\left(1 - \frac{\check{\tau}}{\eta_l \tilde{m}}\right)^2 \frac{\eta_l \tilde{m}}{\check{\tau}} - (n-\tilde{\gamma})\left(\frac{1}{n-1} - \frac{1}{n}\right)\left(\frac{\check{\tau}}{\eta_l \tilde{m}}\right)^n + \frac{\tilde{\gamma}}{n} - \frac{\tilde{\gamma}-1}{n-1} \cdot \frac{\check{\tau}}{\eta_l \tilde{m}} \\ &= (\tilde{\gamma}-1)\left(1 - \frac{\check{\tau}}{\eta_l \tilde{m}}\right)\left(\frac{n}{n-1} - \frac{\eta_l \tilde{m}}{\check{\tau}}\right) + (n-\tilde{\gamma})\left(\frac{1}{n-1} - \frac{1}{n}\right)\left(1 - \left(\frac{\check{\tau}}{\eta_l \tilde{m}}\right)^n\right)\end{aligned} \quad \text{(A24)}$$

and $\Psi_{\tilde{m}}$ is as in (A15). Since $\eta_l \tilde{m}/\check{\tau} < n/(n-1)$ (see Appendix A6.1), $\Psi_{\eta_l} > 0$ given $\check{\tau}/(\eta_l \tilde{m}) < 1$ if $\tilde{\gamma} \le n$. If $\tilde{\gamma} > n$, using $\eta_l \tilde{m}/\check{\tau} < \tilde{\gamma}/(\tilde{\gamma}-1)$, we have

$$\Psi_{\eta_l} > \frac{\tilde{\gamma}-n}{n-1}\left(1 - \frac{\check{\tau}}{\eta_l \tilde{m}} - \frac{1}{n}\left(1 - \left(\frac{\check{\tau}}{\eta_l \tilde{m}}\right)^n\right)\right) > 0 \quad \text{(A25)}$$

given $\check{\tau}/\eta_l \tilde{m} < 1$. Using (A17) and (A23), we have $d\tilde{m}/d\eta_l < 0$.

Appendix A6.3: Proof that $d\check{\tau}/d\eta_l > 0$ and $d(\eta_l\tilde{m})/d\eta_l > 0$ in a Type 2 Solution in Proposition A2

Rewrite Φ in (A8) as a function of η_l, $\check{\tau}$, and $\eta_l\tilde{m}$.

$$\begin{aligned}\Phi(\eta_l,\check{\tau},\eta_l\tilde{m}) &\equiv 1-\frac{\eta_l}{\eta_l\tilde{m}}+\beta\eta_l^{\tilde{\gamma}}\left(\frac{k}{n-1}-\frac{k}{n}\right)\left(\frac{1}{\eta_l\tilde{m}}\right)^n\left(1-\left(\frac{\eta_l}{\eta_h}\right)^{n-\tilde{\gamma}}\right)\\ &\quad -\beta\eta_l^{\tilde{\gamma}}\frac{1}{\check{\tau}^n}\left(\frac{k}{n-1}\cdot\frac{\check{\tau}}{\eta_l\tilde{m}}-\frac{k}{n}\right)\\ &=0.\end{aligned} \tag{A26}$$

We can see that when (A26) holds, Φ becomes negative as η_l rises if $\tilde{\gamma}\le n$, since the sum of the last two terms is negative. If $\tilde{\gamma}>n$, we have

$$\begin{aligned}\frac{\partial\Phi}{\partial\eta_l} &= -\frac{1}{\eta_l\tilde{m}}+\frac{\beta\eta_l^{\tilde{\alpha}}}{\check{\tau}^n}\left(\frac{k}{n(n-1)}\left(\frac{\check{\tau}}{\eta_l\tilde{m}}\right)^n\left(\tilde{\gamma}-n\left(\frac{\eta_l}{\eta_h}\right)^{n-\tilde{\gamma}}\right)-\tilde{\gamma}\left(\frac{k}{n-1}\cdot\frac{\check{\tau}}{\eta_l\tilde{m}}-\frac{k}{n}\right)\right)\\ &< -\frac{1}{\eta_l\tilde{m}}+\frac{\beta\eta_l^{\tilde{\alpha}}}{\check{\tau}^n}\left(\frac{k}{n(n-1)}\left(\frac{\check{\tau}}{\eta_l\tilde{m}}\right)^n(\tilde{\gamma}-n)-\frac{k}{n(n-1)}\cdot\frac{\check{\tau}}{\eta_l\tilde{m}}(\tilde{\gamma}-n)\right.\\ &\qquad\left.-\frac{k}{n-1}\cdot\frac{\check{\tau}}{\eta_l\tilde{m}}+\tilde{\gamma}\frac{k}{n}\left(1-\frac{\check{\tau}}{\eta_l\tilde{m}}\right)\right)\\ &< 0,\end{aligned} \tag{A27}$$

where the first inequality uses $(\eta_l/\eta_h)^{n-\tilde{\gamma}}>1$, and the second inequality uses $\check{\tau}/\eta_l\tilde{m} > (n-1)/n$ and $\check{\tau}/\eta_l\tilde{m} > (\tilde{\gamma}-1)/\tilde{\gamma}$ (see Appendix A6.1). Therefore, when (A26) holds, Φ becomes negative as η_l rises for all $\tilde{\gamma}>1$. We can also see that Φ rises in $\eta_l\tilde{m}$ given $\check{\tau}/(\eta_l\tilde{m})<1$. Then, holding $\check{\tau}$, $\eta_l\tilde{m}$ rises as η_l rises in order for (A26) to hold. Now consider (A26) as defining the optimal $\eta_l\tilde{m}$ as a function of $\check{\tau}$, and consider (1) as defining the optimal $\check{\tau}$ as a function of $\eta_l\tilde{m}$. In the right figure in Figure 3 with the vertical axis $\eta_l\tilde{m}$ instead of $\tilde{m}$, the effect of a rising η_l is to shift up the $\eta_l\tilde{m}$ reaction function around the crossing point raising both the optimal $\check{\tau}$ and the optimal $\eta_l\tilde{m}$. Therefore, we have $d\check{\tau}/d\eta_l>0$ and $d(\eta_l\tilde{m})/d\eta_l>0$.

Appendix A7: The Condition for $d\tilde{m}/d\tilde{\gamma} > 0$ in a Type 2 Solution in Section 4.2

If $\tilde{\eta}_l = \eta_l^{\tilde{\alpha}}$ and $\tilde{\eta}_h = \eta_h^{\tilde{\alpha}}$, differentiating (3) with respect to $\tilde{\gamma}$, we can see that $\tilde{m}(\check{\tau})$ shifts up iff

$$\tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}}(-\log\tilde{\eta}_l)\big(F(\eta_l\tilde{m}) - F(\check{\tau})\big) < \tilde{\eta}_h^{\frac{\tilde{\gamma}}{\tilde{\alpha}}}(\log\tilde{\eta}_h)\big(1 - F(\eta_h\tilde{m})\big). \tag{A28}$$

Intuitively, the optimal $\tilde{m}$ rises iff the fraction of workers overmatched with tasks ($F(\eta_l\tilde{m})$ $- F(\check{\tau})$)) who lose by raising $\tilde{m}$ is small relative to the fraction of workers undermatched with tasks ($1 - F(\eta_h\tilde{m})$) who gain by raising $\tilde{m}$. Condition (A28) is sufficient for both the optimal $\tilde{m}$ and the optimal $\check{\tau}$ to rise as $\tilde{\gamma}$ rises in a type 2 solution. As discussed in the main text, if $\tilde{\gamma} = \bar{\gamma}$, $\check{\tau} = \eta_l\tilde{m}$, so (A28) holds; As $\tilde{\gamma} \to \infty$, (A28) holds as well since $\check{\tau} \to \eta_l\tilde{m} \to \tilde{m} \to \lim_{\tilde{\gamma}\to\infty}\bar{m}$. However, (A28) may not hold for some values of $\tilde{\gamma}$. In Appendix A7.1, assuming the Pareto distribution of talent, I show that $d\tilde{m}/d\tilde{\gamma} > 0$ iff

$$-\tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}}\frac{n\tilde{\alpha}}{\check{\tau}}\left(1-\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)\frac{\partial\check{\tau}}{\partial\tilde{\gamma}} - \tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}}\left(\log\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\right)\left(1-\left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n\right) - \tilde{\eta}_h^{\frac{\tilde{\gamma}}{\tilde{\alpha}}}\left(\log\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\right)\left(\frac{\check{\tau}}{\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n < 0. \tag{A29}$$

where $\partial\check{\tau}/\partial\tilde{\gamma} > 0$ is given by (A31). If $\tilde{\eta}_l = \eta_l^{\tilde{\gamma}}$ and $\tilde{\eta}_h = \eta_h^{\tilde{\gamma}} > 1$, alternative versions of (A28) and (A29) can be derived by following the same steps, with the same interpretations.

In order to assess the plausibility of (A29), consider the case of symmetric talent relation (i.e., $\tilde{\eta}_h\tilde{\eta}_l = 1$). Given $\tilde{\gamma}$, n, and $\tilde{\eta}_l$, we can find the range of $\check{\tau}/(\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m})$ under which (A29) does not hold subject to the lower bound of $\check{\tau}/(\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m})$ given by (A12) and (A16), and use (1) and (A2) to compute the value of the remaining parameter β corresponding to each value of $\check{\tau}/(\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m})$ in the range. For example, consider $n = 2$, which implies that the talent Gini is $1/(2n-1) = 1/3$, which also approximates the income Gini on the right tale of the second-period income distribution, and $\tilde{\gamma} = 2$, which implies that the fixed cost share of the gross output under a uniform talent across tasks is $\tilde{\alpha}/\tilde{\gamma} = 1/2$, as discussed in section 2. With $n = 2$ and $\tilde{\gamma} = 2$, there are no values of $\tilde{\eta}_l$ and β under which (A29) does not hold. With $n = 2$ and $\tilde{\gamma} = 1.5$, the lowest value of β under which (A29) does not hold, is about 18 and requires $\tilde{\eta}_l \approx 0.8$. With $n = 2$ and $\tilde{\gamma} = 1.1$, the lowest value of β under which (A29) does not hold, is about 4.6 and requires $\tilde{\eta}_l \approx 0.99$. In order for (A29) not to hold, a very high value of β (i.e., a very large weight of the second-period income) or a very low value of $\tilde{\gamma}$ (i.e., a very low fixed cost as a fraction of the gross output) coupled with a very high value of $\tilde{\eta}_l$ (i.e., a very small range of tasks across which talent is transferable) is needed.

Appendix A7.1: Proof of Condition (A29)

Given $\eta_l = \tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}$, (1) becomes

$$\check{\tau}(\tilde{m}) = \frac{\tilde{\alpha}}{\tilde{\gamma}} \cdot \tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}} \tilde{m} + \frac{1}{\tilde{\gamma}} \cdot \frac{1}{\tilde{\eta}_l \tilde{m}^{\tilde{\alpha}}}. \tag{A30}$$

We have

$$\frac{\partial \check{\tau}}{\partial \tilde{\gamma}} = \frac{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}} \tilde{m}}{\tilde{\gamma}^2} \left(1 - \left(\frac{1}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}} \tilde{m}}\right)^{\tilde{\gamma}} \left(1 - \tilde{\gamma} \log\left(\frac{1}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}} \tilde{m}}\right) + \tilde{\gamma}\left(\left(\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}} \tilde{m}\right)^{\tilde{\gamma}} - 1\right) \log \tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\right)\right) > 0 \tag{A31}$$

for all $\eta_l \tilde{m} > 1$. Adapting (A13), we have

$$d\check{\tau} = d\tilde{\gamma} \frac{\partial \check{\tau}}{\partial \tilde{\gamma}} + d\tilde{m} \frac{\partial \check{\tau}}{\partial \tilde{m}} = d\tilde{\gamma} \frac{\partial \check{\tau}}{\partial \tilde{\gamma}} + d\tilde{m}(\tilde{\gamma} - 1)\left(1 - \frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}} \tilde{m}}\right) \tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}. \tag{A32}$$

Given $\eta_l = \tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}} = 1/\eta_h$, rewrite Φ in (A8) as a function of $\tilde{\gamma}$, $\check{\tau}$, and $\tilde{m}$.

$$\begin{aligned}
\Phi(\tilde{\gamma}, \check{\tau}, \tilde{m}) \equiv\ & 1 - \frac{1}{\tilde{m}} + \beta \tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}} \left(\frac{k}{n-1} - \frac{k}{n}\right) \left(\frac{1}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}} \tilde{m}}\right)^n \left(1 - \left(\frac{\tilde{\eta}_l}{\tilde{\eta}_h}\right)^{\frac{n-\tilde{\gamma}}{\tilde{\alpha}}}\right) \\
& - \beta \tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}} \frac{1}{\check{\tau}^n} \left(\frac{k}{n-1} \cdot \frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}} \tilde{m}} - \frac{k}{n}\right) \\
=\ & 0.
\end{aligned} \tag{A33}$$

Adapting (A14), we have

$$d\Phi = d\check{\tau} \frac{\partial \Phi}{\partial \check{\tau}} + d\tilde{\gamma} \frac{\partial \Phi}{\partial \tilde{\gamma}} + d\tilde{m} \frac{\partial \Phi}{\partial \tilde{m}} = d\tilde{\gamma} \frac{\beta k}{\check{\tau}^n n \tilde{\alpha}} \cdot \Psi_{\tilde{\gamma}} + d\tilde{m}\left(\frac{\beta \tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}} k}{\check{\tau}^n \tilde{m}} \cdot \tilde{\Psi}_{\tilde{m}} + \frac{1}{\tilde{m}^2}\right) = 0 \tag{A34}$$

where $\partial\Phi/\partial\check{\tau}$ and $\partial\Phi/\partial\tilde{m}$ are as in (A10) and (A11);

$$\frac{\partial\Phi}{\partial\tilde{\gamma}} = \frac{\beta k}{\check{\tau}^n n\tilde{\alpha}}\left(\tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}}\left(\log\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\right)\left(\left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n - 1\right) - \tilde{\eta}_h^{\frac{\tilde{\gamma}}{\tilde{\alpha}}}\left(\log\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\right)\left(\frac{\check{\tau}}{\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n\right); \tag{A35}$$

(A15) becomes

$$\tilde{\Psi}_{\tilde{m}} = -(\tilde{\gamma}-1)\left(1-\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^2\frac{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}{\check{\tau}} + \frac{1}{n-1}\cdot\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\left(1-\left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^{n-1}\left(1-\left(\frac{\tilde{\eta}_l}{\tilde{\eta}_h}\right)^{\frac{n-\tilde{\gamma}}{\tilde{\alpha}}}\right)\right); \tag{A36}$$

and

$$\Psi_{\tilde{\gamma}} = -\tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}}\frac{n\tilde{\alpha}}{\check{\tau}}\left(1-\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)\frac{\partial\check{\tau}}{\partial\tilde{\gamma}} - \tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}}\left(\log\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\right)\left(1-\left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n\right) - \tilde{\eta}_h^{\frac{\tilde{\gamma}}{\tilde{\alpha}}}\left(\log\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\right)\left(\frac{\check{\tau}}{\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n. \tag{A37}$$

Since $\tilde{\Psi}_{\tilde{m}} > 0$, $d\tilde{m}/d\tilde{\gamma} > 0$ in (A34) iff $\Psi_{\tilde{\gamma}} < 0$.

Appendix A8: The Effects of a Rising $\tilde{\gamma}$ on y and y' in Section 4.2

As in section 3, the effect of a rising $\tilde{\gamma}$ on the first period income y is composed of the effect holding $\tilde{m}$, which lowers y, and the effect via $\tilde{m}$, which lowers y iff the optimal $\tilde{m}$ rises. Therefore, as $\tilde{\gamma}$ rises, y falls if $\tilde{m}$ does not fall, but may rise if $\tilde{m}$ falls. As $\tilde{\gamma}$ rises, the second-period income y' stays at one if $\tilde{\tau} < \check{\tau}$ (the first-segment), and $y' = \tilde{\tau}^{\tilde{\gamma}}$ rises if $\tilde{\tau} \in [\eta_l\tilde{m}, \eta_h\tilde{m}]$ (the third segment). If $\tilde{\tau} \in [\check{\tau}, \eta_l\tilde{m})$ (the second segment), given $d(\eta_l\tilde{m})/d\tilde{\gamma} > 0$, we can repeat the steps in Appendix A3 to show that there is $z_1 \in (z_0, \eta_l\tilde{m})$ such that $dy'/d\tilde{\gamma} < 0$ if $\tilde{\tau} \in (\check{\tau}, z_1)$ and $dy'/d\tilde{\gamma} > 0$ if $\tilde{\tau} \in (z_1, \eta_l\tilde{m})$. If $\tilde{\tau} > \eta_h\tilde{m}$ (the fourth segment) and if $\tilde{\eta}_h = \eta_h^{\tilde{\alpha}}$, $y' = \tilde{\tau}\cdot\tilde{\gamma}\tilde{\eta}_h\tilde{m}^{\tilde{\alpha}} - \tilde{\alpha}\tilde{\eta}_h^{\tilde{\gamma}/\tilde{\alpha}}\tilde{m}^{\tilde{\gamma}}$ and

$$\begin{aligned}\frac{\partial y'}{\partial\tilde{\gamma}} &= \tilde{\eta}_h\tilde{m}^{\tilde{\alpha}}\left(\tilde{\tau} - \tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m} + (\tilde{\gamma}\tilde{\tau} - \tilde{\alpha}\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m})\log(\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m}) - \tilde{\gamma}(\log\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}})(\tilde{\tau} - \tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m})\right) \\ &= \tilde{\eta}_h\tilde{m}^{\tilde{\alpha}}\left(\tilde{\tau} - \tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m} + \tilde{\gamma}(\tilde{\tau} - \tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m})\log\tilde{m} + \tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m}\log(\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m})\right) \\ &> 0.\end{aligned} \tag{A38}$$

If $\tilde{\eta}_h = \eta_h^{\tilde{\gamma}}$, $y' = \tilde{\tau}\cdot\tilde{\gamma}^{\frac{\tilde{\alpha}}{\tilde{\gamma}}}\tilde{\eta}_h\tilde{m}^{\tilde{\alpha}} - \tilde{\alpha}\tilde{\eta}_h\tilde{m}^{\tilde{\gamma}}$ and $\partial y'/\partial\tilde{\gamma} > 0$ holds by a variation of (A38). In comparison with the fourth segment in Appendix A3, a higher $\tilde{\gamma}$ raises y' directly and lowers y' by lowering $\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}$, but the former effect dominates the latter effect. The overall effect of a rising $\tilde{\gamma}$ includes the third effect through changing $\tilde{m}$: $dy'/d\tilde{\gamma} = \partial y'/\partial\tilde{\gamma} + (d\tilde{m}/d\tilde{\gamma})(\partial y'/\partial\tilde{m})$. Therefore, $dy'/d\tilde{\gamma} > 0$ unconditionally in a type 1 solution, or if the optimal $\tilde{m}$ does not fall in a type 2 solution. We can also show that $dy'/d\tilde{\gamma} > 0$ regardless of the possibly falling $\tilde{m}$ if the talent is distributed by the Pareto distribution. See Appendix A8.1.

Appendix A8.1: Proof that $dy'/d\tilde{\gamma} > 0$ when $\tilde{\tau} > \eta_h\tilde{m}$ under the Pareto Distribution in Appendix A8

The following proof is under $\tilde{\eta}_l = \eta_l^{\tilde{\alpha}}$ and $\tilde{\eta}_h = \eta_h^{\tilde{\alpha}}$. For the proof under $\tilde{\eta}_l = \eta_l^{\tilde{\gamma}}$ and $\tilde{\eta}_h = \eta_h^{\tilde{\gamma}}$, we can use the steps in Appendix 7.1 and below with minor variation of expressions. The overall effect of rising $\tilde{\gamma}$ is $dy'/d\tilde{\gamma} = \partial y'/\partial\tilde{\gamma} + (d\tilde{m}/d\tilde{\gamma})(\partial y'/\partial\tilde{m})$, which is positive if $d\tilde{m}/d\tilde{\gamma} \geq 0$. Suppose that $d\tilde{m}/d\tilde{\gamma} < 0$. Then, $d\tilde{m}/d\tilde{\gamma} > -(\tilde{m}\Psi_{\tilde{\gamma}})/(\tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}} n\tilde{\alpha}\tilde{\Psi}_{\tilde{m}})$ in (A34), and

$$\begin{aligned}\frac{dy'}{d\tilde{\gamma}} &> \frac{\partial y'}{\partial\tilde{\gamma}} - \frac{\tilde{m}\Psi_{\tilde{\gamma}}}{\tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}} n\tilde{\alpha}\tilde{\Psi}_{\tilde{m}}} \cdot \frac{\partial y'}{\partial\tilde{m}} \\ &= \frac{\partial y'}{\partial\tilde{\gamma}} - \frac{\tilde{m}\Psi_{\tilde{\gamma}}}{\tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}} n\tilde{\alpha}\tilde{\Psi}_{\tilde{m}}} \cdot \tilde{\eta}_h\tilde{m}^{\tilde{\alpha}} \cdot \tilde{\alpha}\tilde{\gamma}\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\left(\frac{\tilde{\tau}}{\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m}} - 1\right).\end{aligned} \tag{A39}$$

When $\tilde{\tau}(\tilde{m}) = \tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m}$, $dy'/d\tilde{\gamma} > 0$, so $dy'/d\tilde{\gamma} > 0$ for all $\tilde{\tau} \geq \tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m}$ if the derivative of the righthand side of (A39) with respect to $\tilde{\tau}$ is not negative; that is,

$$-\frac{\Psi_{\tilde{\gamma}}}{\tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}} n\tilde{\Psi}_{\tilde{m}}} \geq -\frac{1}{\tilde{\gamma}} - \log\tilde{m} = -\frac{1}{\tilde{\gamma}} - \log\left(\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}\right) + \log\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}. \tag{A40}$$

Using (A31),(A37), and (A12), we have

$$\begin{aligned}-\frac{\Psi_{\tilde{\gamma}}}{\tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}} n} = &\frac{\tilde{\alpha}}{\tilde{\gamma}}\left(\frac{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}{\check{\tau}} - 1\right)\left(1 - \frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}} + \left(-\tilde{\alpha} + \tilde{\gamma}\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)\log\left(\frac{1}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right) - \tilde{\gamma}\left(1 - \frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)\log\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\right) \\ &+ \frac{1}{n}\left(\log\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\right)\left(1 - \left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n\right) + \frac{1}{n}\left(\frac{\tilde{\eta}_h}{\tilde{\eta}_l}\right)^{\frac{\tilde{\gamma}}{\tilde{\alpha}}}\left(\log\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\right)\left(\frac{\check{\tau}}{\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n.\end{aligned} \tag{A41}$$

Using (A41) and (A36), after some algebra, we can write (A40) as

$$A \log\left(\frac{1}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right) + B \log \tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}} + C \geq 0 \tag{A42}$$

where

$$A \equiv \frac{\tilde{\alpha}}{\tilde{\gamma}}\left(\frac{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}{\check{\tau}} - 1\right) - \frac{1}{n-1}\left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}} - \left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n\left(1 - \left(\frac{\tilde{\eta}_l}{\tilde{\eta}_h}\right)^{\frac{n-\tilde{\gamma}}{\tilde{\alpha}}}\right)\right), \tag{A43}$$

$$B \equiv \frac{1}{n} - \frac{1}{n}\left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n - \frac{1}{n-1}\left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}} - \left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n\left(1 - \left(\frac{\tilde{\eta}_l}{\tilde{\eta}_h}\right)^{\frac{n-\tilde{\gamma}}{\tilde{\alpha}}}\right)\right), \tag{A44}$$

and

$$C \equiv \frac{1}{n}\left(\frac{\tilde{\eta}_h}{\tilde{\eta}_l}\right)^{\frac{\tilde{\gamma}}{\tilde{\alpha}}}\left(\log \tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\right)\left(\frac{\check{\tau}}{\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n + \frac{1}{\tilde{\gamma}(n-1)}\left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}} - \left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n\left(1 - \left(\frac{\tilde{\eta}_l}{\tilde{\eta}_h}\right)^{\frac{n-\tilde{\gamma}}{\tilde{\alpha}}}\right)\right) > 0. \tag{A45}$$

Using (A16), we have

$$A < \frac{\tilde{\alpha}}{\tilde{\gamma}}\frac{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}{\check{\tau}}\left(1 - \frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right) - \frac{1}{n-1}\cdot\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}} + \frac{n}{n-1}\cdot\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}} - 1 < 0 \tag{A46}$$

where the last inequality uses $(\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m})/\check{\tau} < \tilde{\gamma}/\tilde{\alpha}$ (see Appendix A6.1), and

$$B < \frac{1}{n} - \frac{1}{n}\left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n - \frac{1}{n-1}\cdot\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}} + \frac{n}{n-1}\cdot\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}} - 1 < 0. \tag{A47}$$

Therefore, (A40) holds and $dy'/d\tilde{\gamma} > 0$ for all $\tilde{\tau} > \tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m}$.

Appendix A9: The Effects of a Rising $\tilde{\gamma}$ on the Constrained $\check{\tau}$ in Section 4.3

If $\eta_l = \tilde{\eta}_l^{\frac{1}{\tilde{\gamma}}}$, (A4) is valid with Ω replaced by

$$\tilde{\Omega}_{\tilde{\gamma}} \equiv \Omega + \tilde{\alpha}(1 - (\tilde{\eta}_l^{\frac{1}{\tilde{\gamma}}}\tilde{m})^{-\tilde{\gamma}})(-\log \tilde{\eta}_l^{\frac{1}{\tilde{\gamma}}}). \tag{A48}$$

The additional term reflects the positive effect of a rising η_l. Taking the same steps as in (A5), we have $d\check{\tau}/d\tilde{\gamma} < 0$.

If $\eta_l = \tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}$, (A4) is valid with Ω replaced by

$$\tilde{\Omega} \equiv \Omega + \tilde{\gamma}(1 - (\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m})^{-\tilde{\gamma}})(-\log \tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}). \tag{A49}$$

We have $\tilde{\Omega} = 0$ if $\eta_l = 1/\tilde{m}$. We can derive

$$\frac{\partial \tilde{\Omega}}{\partial \eta_l} = \frac{\tilde{\gamma}\tilde{\eta}_l^{-\frac{1}{\tilde{\alpha}}}}{\tilde{m} - 1}\bigg(- (\tilde{m} - 1) + (\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m})^{-\tilde{\gamma}}\tilde{m}\log \tilde{m}\bigg). \tag{A50}$$

We can show that the expression inside the large bracket is positive when $\eta_l = 1/\tilde{m}$ given $\tilde{m} > 1$, falls as η_l rises, and becomes zero when $\eta_l = \hat{\eta}_l(\tilde{m}) \equiv (\log \tilde{m}/(\tilde{m}^{\tilde{\alpha}}(\tilde{m} - 1)))^{1/\tilde{\gamma}}$. For a constrained worker, therefore, a sufficient condition for $d\check{\tau}/d\tilde{\gamma} > 0$ is $\eta_l \in (1/\tilde{m}, \hat{\eta}_l(\tilde{m}))$. The bound $\hat{\eta}_l(\tilde{m})$ falls from 1 to 0 as $\tilde{m}$ rises from 1 to ∞. Then, given $\tilde{\gamma} > 1$, an alternative expression of the sufficient condition for $d\check{\tau}/d\tilde{\gamma} > 0$ is $\tilde{m} \in (1/\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}, \hat{\eta}_l^{-1}(\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}))$. If $\tilde{\gamma} \leq \bar{\gamma}$, $\tilde{m} \leq 1/\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}$ for any ω, so the condition $\tilde{m} \in (1/\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}, \hat{\eta}_l^{-1}(\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}))$ does not hold.

In order to proceed further, let $\bar{\omega}(\tilde{\gamma})$ denote the threshold ω for which the first-period income constraint is just non-binding given $\tilde{\gamma}$: $y = \bar{\omega}(\tilde{\gamma})$ under the unconstrained optimal $\tilde{m}$ given $\tilde{\gamma}$. Let $\hat{\omega}(\tilde{\gamma})$ denote the first-period income y given $\tilde{m} = 1/\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}$: $\hat{\omega}(\tilde{\gamma}) = \tilde{\gamma}/\tilde{\eta}_l - \tilde{\alpha}/\tilde{\eta}_l^{\tilde{\gamma}/\tilde{\alpha}}$. We have $\bar{\omega}(\bar{\gamma}) = \hat{\omega}(\bar{\gamma})$ and $\bar{\omega}(\tilde{\gamma}) < \hat{\omega}(\tilde{\gamma})$ for $\tilde{\gamma} > \bar{\gamma}$ since the unconstrained optimal $\tilde{m} > 1/\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}$ for $\tilde{\gamma} > \bar{\gamma}$. Therefore, for $\tilde{\gamma} > \bar{\gamma}$, $(\bar{\omega}(\tilde{\gamma}), \hat{\omega}(\tilde{\gamma}))$ defines the range of ω under which the worker is constrained and the constrained $\check{\tau} > 1$. We can show that $d\hat{\omega}(\tilde{\gamma})/d\tilde{\gamma} > 0$ for all $\tilde{\gamma} \geq \bar{\gamma}$ and $\hat{\omega}(\tilde{\gamma}) \to (1 + \log \tilde{\eta}_l)/\tilde{\eta}_l < 1$ as $\tilde{\gamma} \to \infty$. Then, if $\omega \geq (1 + \log \tilde{\eta}_l)/\tilde{\eta}_l$, $\check{\tau} = 1$ for all $\tilde{\gamma} > 1$. If $\omega \in (\bar{\omega}(\bar{\gamma}), (1 + \log \tilde{\eta}_l)/\tilde{\eta}_l)$, $\check{\tau} = 1$ for $\tilde{\gamma} \leq \hat{\omega}^{-1}(\omega)$; $\check{\tau} > 1$ for $\tilde{\gamma} > \hat{\omega}^{-1}(\omega)$; and $\check{\tau} \to 1$ as $\tilde{\gamma} \to \infty$, where the last property holds since for any $\omega < 1$, the worker becomes constrained with the constrained $\tilde{m} \to 1$ as $\tilde{\gamma} \to \infty$. If $\omega \leq \bar{\omega}(\bar{\gamma})$, $\check{\tau} = 1$ for $\tilde{\gamma} \leq \bar{\gamma}$; $\check{\tau} > 1$ for $\tilde{\gamma} > \bar{\gamma}$; and $\check{\tau} \to 1$ as $\tilde{\gamma} \to \infty$. Further, the above property of $\tilde{\Omega}$ implies that given $\tilde{\gamma} > \bar{\gamma}$, there is an $\epsilon > 0$ such that $d\check{\tau}/d\tilde{\gamma} > 0$ if $\omega \in (\hat{\omega}(\tilde{\gamma}) - \epsilon, \hat{\omega}(\tilde{\gamma}))$. It also implies that given $\tilde{\gamma} > 1$, there is an $\epsilon > 0$ such that $d\check{\tau}/d\tilde{\gamma} > 0$ for a constrained worker with $\check{\tau} \in (1, 1+\epsilon)$.

Appendix A10: The Effects of a Rising $\tilde{\gamma}$ on the Constrained y' in Section 4.3

If $\eta_l = \tilde{\eta}_l^{\frac{1}{\tilde{\gamma}}}$, the constrained $\check{\tau}$ falls as $\tilde{\gamma}$ rises (see Appendix A9), so y' stays constant in the first segment ($\tilde{\tau} < \check{\tau}$), and rises in the second and the third segments ($\tilde{\tau} \in [\check{\tau}, \eta_h\tilde{m}]$) as in section 3.1. In the fourth segment ($\tilde{\tau} > \eta_h\tilde{m}$), a rise of $\tilde{\gamma}$ lowers a constrained worker's $\eta_h\tilde{m}$ by lowering $\tilde{m}$ as in section 3.1 and also by lowering η_h. Nonetheless, y' rises. In order to see this, note that in the fourth segment, $y' = \tilde{\tau} \cdot \tilde{\gamma}\eta_h\tilde{m}^{\tilde{\alpha}} - \tilde{\alpha}\tilde{\eta}_h^{\tilde{\gamma}/\tilde{\alpha}}\tilde{m}^{\tilde{\gamma}}$ and rises when $\tilde{\tau} = \eta_h\tilde{m}$. By (A6), $d(\tilde{\gamma}\tilde{m}^{\tilde{\alpha}})/d\tilde{\gamma} > 0$ for all $\tilde{m} > 1$. Then, $y'|_{\tilde{\tau}>\eta_h\tilde{m}} - y'|_{\tilde{\tau}=\eta_h\tilde{m}}$ rises as $\tilde{\gamma}$ rises. Therefore, y' rises for all $\tilde{\tau} > \eta_h\tilde{m}$, as when $\tilde{\gamma}$ was rising without changing η_l and η_h in section 3.1.

If $\eta_l = \tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}$, as $\tilde{\gamma}$ rises, y' stays constant in the first segment ($\tilde{\tau} < \check{\tau}$), and rises in the third and fourth segments ($\tilde{\tau} > \eta_l\tilde{m}$) as when $\eta_l = \tilde{\eta}_l^{\frac{1}{\tilde{\gamma}}}$ above: Repeat the same steps of reasoning for the fourth segment. In the second segment, y' may fall as $\tilde{\gamma}$ rises: For a constrained worker whose $\check{\tau}$ is above but close enough to one, $\check{\tau}$ rises as $\tilde{\gamma}$ rises (see Appendix A9), lowering y' toward one for $\tilde{\tau}$ above but close enough to $\check{\tau}$. This result is qualitatively the same as for the unconstrained worker in sections 3 and 4.2. Since $\check{\tau}$ falls back toward one as $\tilde{\gamma}$ continues to rise, however, holding $\tilde{\tau}$ this fall is reversed and y' rises as $\tilde{\gamma}$ continues to rise.

Appendix A11: The Effects of a Rising ρ in Section 5

Given $y = \tau(m)^\rho \cdot am^\alpha - bm^\gamma$, we can write $y = \tilde{\tau}(\tilde{m})^\rho \cdot \tilde{\gamma}\tilde{m}^{\rho\tilde{\alpha}} - \tilde{\alpha}\tilde{m}^{\rho\tilde{\gamma}}$ where $\tilde{m} \equiv (m^{\gamma-\alpha}(b/a)(\tilde{\gamma}/\tilde{\alpha}))^{1/\rho}$, $\tilde{\tau}(\tilde{m}) \equiv \tau((\tilde{m}^\rho(a/b)(\tilde{\alpha}/\tilde{\gamma}))^{1/(\gamma-\alpha)})$, and $\Upsilon \equiv (\tilde{\alpha}/b)^{\tilde{\alpha}}(a/\tilde{\gamma})^{\tilde{\gamma}} = 1$ by normalization. If $\tilde{\tau}(\tilde{m})$ is constant at τ for all $\tilde{m}$, the maximum y is $\tau^{\rho\tilde{\gamma}}$ obtained by choosing $\tilde{m} = \tau$. The maximum obtainable income is $\bar{y} = \max_{\tilde{m}}\{E[\tau^\rho] \cdot \tilde{\gamma}\tilde{m}^{\rho\tilde{\alpha}} - \tilde{\alpha}\tilde{m}^{\rho\tilde{\gamma}})\} = (E[\tau^\rho])^{\tilde{\gamma}}$. We will assume that talent uncertainty for the first-period task is fully resolved after one period, so we can write $E[\tilde{\tau}(\tilde{m})^\rho] = \tilde{\tau}(\tilde{m})^\rho$ for the first-period task $\tilde{m}$ after one period. However, the initial expectation of effective talent $E[\tilde{\tau}^\rho] = \int_0^\infty \tilde{\tau}^\rho dF(\tilde{\tau}) \neq (\int_0^\infty \tilde{\tau} dF(\tilde{\tau}))^\rho = E[\tilde{\tau}]^\rho$ generally. Relations (1) to (4) are replaced by

$$\check{\tau}^\rho = \frac{(E[\tilde{\tau}^\rho])^{\tilde{\gamma}}}{\tilde{\gamma}(\eta_l\tilde{m})^{\rho\tilde{\alpha}}} + \frac{\tilde{\alpha}(\eta_l\tilde{m})^\rho}{\tilde{\gamma}}, \tag{A51}$$

$$\int_0^\infty (\tilde{m}^\rho - \tilde{\tau}^\rho)dF(\tilde{\tau}) < \beta\eta_h^{\rho\tilde{\alpha}} \int_{\eta_h\tilde{m}}^\infty (\tilde{\tau}^\rho - (\eta_h\tilde{m})^\rho)dF(\tilde{\tau}), \tag{A52}$$

$$\int_0^\infty (\tilde{m}^\rho - \tilde{\tau}^\rho)dF(\tilde{\tau}) + \beta\eta_l^{\rho\tilde{\alpha}} \int_{\check{\tau}}^{\eta_l\tilde{m}} ((\eta_l\tilde{m})^\rho - \tilde{\tau}^\rho)dF(\tilde{\tau}) < \beta\eta_h^{\rho\tilde{\alpha}} \int_{\eta_h\tilde{m}}^\infty (\tilde{\tau}^\rho - (\eta_h\tilde{m})^\rho)dF(\tilde{\tau}), \tag{A53}$$

and

$$\frac{1}{\eta_l^\rho} - 1 \geq \beta\eta_h^{\rho\tilde{\alpha}} \int_{(\eta_h/\eta_l)(E[\tau^\rho])^{1/\rho}}^\infty \left(\frac{\tilde{\tau}^\rho}{E[\tau^\rho]} - \frac{\eta_h^\rho}{\eta_l^\rho}\right) dF(\tilde{\tau}). \tag{A54}$$

In (A51), $\check{\tau}$ rises in ρ. Write (A53) with equality as

$$\int_0^\infty \left(1 - \left(\frac{\tilde{\tau}}{\tilde{m}}\right)^\rho\right) dF(\tilde{\tau}) + \beta\eta_l^{\rho\tilde{\gamma}} \int_{\check{\tau}}^{\eta_l\tilde{m}} \left(1 - \left(\frac{\tilde{\tau}}{\eta_l\tilde{m}}\right)^\rho\right) dF(\tilde{\tau}) - \beta\eta_h^{\rho\tilde{\gamma}} \int_{\eta_h\tilde{m}}^\infty \left(\left(\frac{\tilde{\tau}}{\eta_h\tilde{m}}\right)^\rho - 1\right) dF(\tilde{\tau}) = 0. \tag{A55}$$

The lefthand side is the marginal cost of raising $\tilde{m}$ net of the marginal benefit, divided by a common factor. The factors in (A55) can be interpreted in terms of mismatch factors and mismatch factor weights in the same fashion as the factors in (A53). The overmatch factor weight $\beta\eta_l^{\rho\tilde{\gamma}}$ falls in ρ, and the undermatch factor weight $\beta\eta_h^{\rho\tilde{\gamma}}$ rises in ρ, lowering the net marginal cost. Let $X \equiv \int_0^\infty (1 - (\tilde{\tau}/\tilde{m})^\rho) dF(\tilde{\tau})$, $Y \equiv \int_{\check{\tau}}^{\eta_l\tilde{m}} (1 - (\tilde{\tau}/\eta_l\tilde{m})^\rho) dF(\tilde{\tau})$, and $Z \equiv \int_{\eta_h\tilde{m}}^\infty ((\tilde{\tau}/\eta_h\tilde{m})^\rho - 1) dF(\tilde{\tau})$ denote the three mismatch factors. We have $\rho((\partial X/\partial\rho) + \beta\eta_l^{\rho\tilde{\gamma}}(\partial Y/\partial\rho) - \beta\eta_h^{\rho\tilde{\gamma}}(\partial Y/\partial\rho)) < X + \beta\eta_l^{\rho\tilde{\gamma}}Y - \beta\eta_h^{\rho\tilde{\gamma}}Y = 0$: As ρ rises, changes in mismatch factors lower the net marginal cost. Therefore, a rising ρ lowers the net marginal cost, raising $\tilde{m}$ in (A55). Following the same steps in (A52), we have $\tilde{m}$ rising in ρ. In Figure 3, a rising ρ shifts the $\tilde{m}$ reaction function up and shifts the $\check{\tau}$ reaction function to the right, raising the optimal $\tilde{m}$ and, in the case of a type 2 solution, raising the optimal $\check{\tau}$. In

(A54), a rising ρ can turn a type 2 solution to a type 1 solution in contrast with a rising $\tilde{\gamma}$.

Now consider the bounds of the task range given by a fixed ratio of talent productivity or production cost as in section 4: $\tilde{\eta}_h = \eta_h^\rho > 1$ and $\tilde{\eta}_l = \eta_l^\rho < 1$ are held constant. Then, η_h falls and η_l rises in ρ. In (A54), the righthand side rises in ρ, so a rising ρ can turn a type 1 solution to a type 2 solution but not the other way around. In (A52) and (A53), mismatch factor weights $\eta_l^{\rho\tilde{\alpha}}$ and $\eta_h^{\rho\tilde{\alpha}}$ are held constant while the undermatch factor $\int_{\eta_h\tilde{m}}^\infty (\tilde{\tau}^\rho - (\eta_h\tilde{m})^\rho)dF(\tilde{\tau})$ rises as η_h falls, and the overmatch factor $\int_{\check{\tau}}^{\eta_l\tilde{m}}((\eta_l\tilde{m})^\rho - \tilde{\tau}^\rho)dF(\tilde{\tau})$ rises as η_l rises. The latter effect cancels the negative effect of ρ on η_l^ρ that was present when η_l was held constant. If this latter effect is strong enough, the optimal $\tilde{m}$ falls. In (A55), holding $\tilde{m}$, the net marginal cost eventually becomes negative as ρ continues to rise. Therefore, even when the optimal $\tilde{m}$ falls, it eventually rises to a higher value as ρ continues to rise. Overall, a rising ρ raises the optimal task levels with qualifications as a rising $\tilde{\gamma}$ does. One difference is that, as can be seen in (A55), the optimal $\tilde{m}$ rises without a bound even when η_h^ρ and η_l^ρ are held constant.

Unlike a rising $\tilde{\gamma}$, a rising ρ raises the current-income maximizing task level $\tilde{m} = (E[\tilde{\tau}^\rho])^{1/\rho}$ and changes the associated maximum current income $\bar{y} = (E[\tilde{\tau}^\rho])^{\tilde{\gamma}}$. This implies that the responses of career-based task choices and incomes to a rising ρ are ambiguous. Formally, define career-based tasks as multiples of the current-income maximizing task: $\tilde{M} \equiv \tilde{m}/E[\tilde{\tau}^\rho]^{1/\rho}$, $\tilde{T} \equiv \tilde{\tau}/E[\tilde{\tau}^\rho]^{1/\rho}$, and $G(\tilde{T}) \equiv F(\tilde{T}E[\tilde{\tau}^\rho]^{1/\rho})$. With these changes of variables, (A51) to (A54) become

$$\check{T}^\rho = \frac{1}{\tilde{\gamma}(\eta_l\tilde{M})^{\rho\tilde{\alpha}}} + \frac{\tilde{\alpha}(\eta_l\tilde{M})^\rho}{\tilde{\gamma}}, \tag{A56}$$

$$\tilde{M}^\rho - 1 < \beta\eta_h^{\rho\tilde{\alpha}} \int_{\eta_h\tilde{M}}^\infty (\tilde{T}^\rho - (\eta_h\tilde{M})^\rho)dG(\tilde{T}), \tag{A57}$$

$$\tilde{M}^\rho - 1 + \beta\eta_l^{\rho\tilde{\alpha}} \int_{\check{T}}^{\eta_l\tilde{M}} ((\eta_l\tilde{M})^\rho - \tilde{T}^\rho)dG(\tilde{T}) < \beta\eta_h^{\rho\tilde{\alpha}} \int_{\eta_h\tilde{M}}^\infty (\tilde{T}^\rho - (\eta_h\tilde{M})^\rho)dG(\tilde{T}), \tag{A58}$$

and

$$\frac{1}{\eta_l^\rho} - 1 \geq \beta\eta_h^{\rho\tilde{\alpha}} \int_{\eta_h/\eta_l}^{\infty} \left(\tilde{T}^\rho - \frac{\eta_h^\rho}{\eta_l^\rho}\right) dG(\tilde{T}). \tag{A59}$$

As ρ rises, G changes possibly countering the positive effect of a rising ρ on $\tilde{M}$ discussed above. Therefore, the overall effect of a rising ρ on $\tilde{M}$ is ambiguous and the effects on incomes as multiples of the maximum current income, $y/\bar{y} = \tilde{\gamma}\tilde{M}^{\tilde{\alpha}} - \tilde{\alpha}\tilde{M}^{\tilde{\gamma}}$ and $y'/\bar{y} = \tilde{T} \cdot \tilde{\gamma}(\tilde{M}')^{\tilde{\alpha}} - \tilde{\alpha}(\tilde{M}')^{\tilde{\gamma}}$, are also ambiguous.

Appendix A12: Conditions for a Negative First-period Optimal Income y in Section 5

We have $y = \tilde{\gamma}\tilde{m}^{\tilde{\alpha}} - \tilde{\alpha}\tilde{m}^{\tilde{\gamma}} < 0$ iff $\tilde{m} > \tilde{\gamma}/\tilde{\alpha}$. As $\tilde{\gamma} \to \infty$, $\tilde{\gamma}/\tilde{\alpha} \to 1$, so y becomes negative eventually as $\tilde{\gamma}$ rises as long as $\lim_{\tilde{\gamma}\to\infty} \tilde{m} > 1$, which holds in both sections 3 and 4.2. For the remainder, assume the Pareto distribution of talent (assumption 3-2). Suppose $\eta_l \leq \tilde{\alpha}/\tilde{\gamma}$. Then, $\eta_l\tilde{m} \leq 1$ and $\check{\tau} = 1$ when $\tilde{m} = \tilde{\gamma}/\tilde{\alpha}$. Then, the optimal $\tilde{m} > \tilde{\gamma}/\tilde{\alpha}$ and $y < 0$ iff (A1) holds when $\tilde{m} = \tilde{\gamma}/\tilde{\alpha}$, which, given $\bar{\tau} = n/(n-1) \cdot (k/n)^{1/n} = 1$, is equivalent to

$$\left(\frac{n}{\tilde{\gamma}-1}\right)^n \left(\frac{\tilde{\gamma}}{n-1}\right)^{n-1} < \frac{\beta}{\eta_h^{n-\tilde{\gamma}}}. \tag{A60}$$

Now suppose $\eta_l > \tilde{\alpha}/\tilde{\gamma}$. Then, $\eta_l\tilde{m} > 1$ and $\check{\tau} > 1$ when $\tilde{m} = \tilde{\gamma}/\tilde{\alpha}$. Then, the optimal $\tilde{m} > \tilde{\gamma}/\tilde{\alpha}$ and $y < 0$ iff (A2) holds when $\tilde{m} = \tilde{\gamma}/\tilde{\alpha}$ and $\check{\tau}$ is given by (1), which is equivalent to

$$\left(\frac{n}{\tilde{\gamma}-1}\right)^n \left(\frac{\tilde{\gamma}}{n-1}\right)^{n-1} < \frac{\beta}{\eta_h^{n-\tilde{\gamma}}} - \frac{\beta}{\eta_l^{n-\tilde{\gamma}}} \left(1 - \frac{1-n+n(1-(1/\tilde{\gamma})(1-(1/\eta_l)^{\tilde{\gamma}}(\tilde{\alpha}/\tilde{\gamma})^{\tilde{\gamma}}))}{(1-(1/\tilde{\gamma})(1-(1/\eta_l)^{\tilde{\gamma}}(\tilde{\alpha}/\tilde{\gamma})^{\tilde{\gamma}}))^n}\right). \tag{A61}$$

In order to assess the plausibility of (A60) and (A61), consider $n = 2$ and $\tilde{\gamma} = 2$ used in Appendix A7, and $\eta_l \leq 1/2$ so that (A60) is the relevant condition. Then, (A60) becomes $\beta > 8$; that is, the weight of the second period income must be more than eight times the weight of the first-period income. If $\tilde{\gamma}$ rises to 3 keeping $\eta_l \leq 1/2$, (A60) becomes $\beta > 3/\eta_h$. The threshold value of β is now less than three, and falls below one if $\eta_h > 3$.

Appendix B: A Three-Period Model with a Numerical Exercise

There are three periods, denoted by $t = 0, 1, 2$. There are at least three trades, indexed by $s \geq 0$. The trades are ordered by talent rewards: $\tilde{\gamma}^s \geq \tilde{\gamma}^{s'}$ for $s < s'$. The bounds of the talent-update task range η_l^s and η_h^s may depend on $\tilde{\gamma}^s$ as will be discussed shortly. Trades are equivalent in all other aspects. In particular, the scale factor $\Upsilon^s = 1$ for all s so that given $\bar{\tau} = 1$, the unconditional maximum income $\bar{y}^s = \bar{\tau}^{\tilde{\gamma}^s} \cdot \Upsilon^s = 1$ for all s. The worker's talent across tasks evolves as follows. Let $\tau_t(\tilde{m}, s)$ denote the worker's (expected) talent for task $\tilde{m}$ in trade s at the beginning of t. Let $\tilde{m}_t$ and s_t denote the worker's task and trade at t. At the beginning of $t = 0$, $\tau_0(\tilde{m}, s) = \bar{\tau} = 1$ for all $(\tilde{m}, s)$. At the beginning of $t = 1$, the worker draws $\tau_1(\tilde{m}_0, s_0)$ from the distribution $F_0(\tau)$ with $E[\tau] = \bar{\tau} = 1$; $\tau_1(\tilde{m}, s_0) = \tau_1(\tilde{m}_0, s_0)$ if $\tilde{m} \in [\eta_l^{s_0}\tilde{m}_0, \eta_h^{s_0}\tilde{m}_0]$; and $\tau_1(\tilde{m}, s) = \bar{\tau} = 1$ if $s \neq s_0$ or if $s = s_0$ and $\tilde{m} \notin [\eta_l^{s_0}\tilde{m}_0, \eta_h^{s_0}\tilde{m}_0]$. At the beginning of $t = 2$, the worker draws $\tau_2(\tilde{m}_1, s_1)$ from the distribution $F_0(\tau)$ if $s_1 \neq s_0$ or if $s_1 = s_0$ and $\tilde{m}_1 \notin [\eta_l^{s_0}\tilde{m}_0, \eta_h^{s_0}\tilde{m}_0]$, and from the distribution $F_1(\tau|\tau_1(\tilde{m}_1, s_1))$ with $E[\tau] = \tau_1(\tilde{m}_1, s_1)$ if $s_1 = s_0$ and $\tilde{m}_1 \in [\eta_l^{s_0}\tilde{m}_0, \eta_h^{s_0}\tilde{m}_0]$; $\tau_2(\tilde{m}, s_1) = \tau_2(\tilde{m}_1, s_1)$ if $\tilde{m} \in [\eta_l^{s_1}\tilde{m}_1, \eta_h^{s_1}\tilde{m}_1]$, and $\tau_2(\tilde{m}, s) = \bar{\tau} = 1$ if $s \neq s_1$ or if $s = s_1$ and $\tilde{m} \notin [\eta_l^{s_1}\tilde{m}_1, \eta_h^{s_1}\tilde{m}_1]$. In comparison with the two-period model, the (expected) talent for a task is subject to a further update, distributed by F_1, following the initial update, distributed by F_0.

The worker's trade is assumed to evolve as follows. The worker starts at trade 0 at $t = 0$ and, if he moves to another trade at $t = 1$, he moves to trade 1. If the worker moves to yet another trade at $t = 2$, he moves to any trade $s \neq s_1$ and obtains $\bar{y}^s = 1$. This pattern of the worker's trade would be without a loss of generality if the worker's expected lifetime income rises by raising $\tilde{\gamma}^{s_0}$ or $\tilde{\gamma}^{s_1}$ and, holding $\{s_0, s_1\}$, if the worker's expected lifetime income is higher when working in the trade with a higher $\tilde{\gamma}^s$ first. Since the upside income potential is higher with a higher $\tilde{\gamma}^s$, this property is plausible and appears to hold in the numerical exercise conducted. Note that the worker has three options at the beginning of $t = 1$. The worker can stay in trade 0 and attempt $\tilde{m} \in [\eta_l^0 \tilde{m}_0, \eta_h^0 \tilde{m}_0]$ (option 1). The worker can stay in trade 0 and attempt $\tilde{m} \notin [\eta_l^0 \tilde{m}_0, \eta_h^0 \tilde{m}_0]$ (option 2). The worker can move to trade 1 (option 3). In the two-period model, staying in the current trade and attempting a task outside the talent-update task range (option 2) was not viable since it was (weakly) dominated by moving to another trade and obtaining the unconditional maximum income $\bar{y} = 1$. In the three-period model, option 2 can dominate moving to trade 1 (option 3) as will become clear.

In the numerical exercise, I assume that $F_0(\tau)$ is a bounded Pareto distribution with the support $[1/\sigma_0, \sigma_0]$ where $\sigma_0 > 1$, and the shape parameter $n = 1/2$, which ensures that $E[\tau] = 1$. Similarly, $F_1(\tau|\tau')$ is a bounded Pareto distribution with the support $[\tau'/\sigma_1, \tau'\sigma_1]$ and the shape parameter $n = 1/2$, which ensures that $E[\tau|\tau'] = \tau'$. I set $\sigma_0 = 2$, which implies that the talent Gini at the beginning of $t = 1$ is $1 - 2/(\sigma_0 - 1) \cdot (\sigma_0/(\sigma_0 - 1) \cdot \log \sigma_0 - 1) = 0.23$ given $n = 1/2$. I also set $\sigma_1 = \sigma_0^\nu$ where $\nu \in [0, 1]$, which implies that the expected size of talent update becomes (weakly) smaller over time. Unlike in the two-period model, talent rewards at the second trade $\tilde{\gamma}^1$ affect the career paths. I set $\tilde{\gamma}^1 = \rho\tilde{\gamma}^0 + (1 - \rho) \cdot 1.5$ where $\rho \in [0, 1]$. This implies that $\tilde{\gamma}^1 = \tilde{\gamma}^0$ when $\tilde{\gamma}^0 = 1.5$ regardless of ρ. The value of $\tilde{\gamma} = 1.5$ implies that the fixed cost share of the gross output is $\tilde{\alpha}/\tilde{\gamma} = (\tilde{\gamma} - 1)/\tilde{\gamma} = 1/3$. I explore the changes of the worker's career paths when $\tilde{\gamma}^0$ rises from 1.5. If $\rho = 0$, $\tilde{\gamma}^1$ stays at 1.5 as $\tilde{\gamma}^0$ rises. If $\rho = 1$, as $\tilde{\gamma}^0$ rises, $\tilde{\gamma}^1$ rises as much as $\tilde{\gamma}^0$. A low value of ρ implies that rising talent rewards affect a narrow range of trades from the perspective of the worker so that when the worker fails in the initial trade, he moves to a trade with a significantly slow growth of talent rewards. A high value of ρ implies the opposite. I also parameterize the degree of rising task differentiation as $\tilde{\gamma}^s$ rises: $\eta_h^s = 1/\eta_l^s = 2^{\mu/(2\tilde{\alpha}^s)+1-\mu}$ for all s, where $\mu \in [0, 1]$. This implies that when $\tilde{\gamma}^1 = \tilde{\gamma}^0 = 1.5$, $\eta_h^s = 1/\eta_l^s = 2$. If $\mu = 0$, η_h^s stays at 2 as $\tilde{\gamma}_s$ rises as in section 3. If $\mu = 1$, $\eta_h^s = (2^{1/2})^{1/\tilde{\alpha}^s}$ falls as $\tilde{\gamma}_s$ rises as it does under the fixed ratio of talent productivity in section 4. I set the discount parameter $\beta = 1$: there are no discounts for the second and the third period incomes.

Tables 1 and 2 present the numerical results under $\rho = 0$ and $\rho = 1$, respectively, with $\nu = 1/2$ in both cases. In each table, rows 4 to 16 present results when the worker is not constrained while rows 17 to 29 present results when the worker must earn at least 85 percent of the unconditional maximum income in the first period ($y_0 \geq \omega = 0.85$) and a double of that amount over the first two periods ($y_0 + y_1 \geq 2\omega = 1.70$). This is equivalent to assuming that the worker holds no initial wealth, must spend at least 85 percent of the unconditional maximum income in each period, cannot borrow, but can save at the zero interest rate. The value of $\omega = 0.85$ was chosen to construct the narrative that as $\tilde{\gamma}^0$ rises, the income constraint is not binding for a moderate rise but is binding eventually as $\tilde{\gamma}^0$ continues to rise.

<u>Benchmark: $\tilde{\gamma}^0 = 1.5$ and any values of μ and ρ</u>

Column 2 presents results when $\tilde{\gamma}^0 = 1.5$, in which case the values of μ and ρ do not matter. Under $\tilde{\gamma}^0 = 1.5$, the interval $[\eta_l^0, \eta_h^0] = [0.5, 2]$ includes the intervals $[1/\sigma_0, \sigma_0] = [0.5, 2]$ and $[1/\sigma_1, \sigma_1] \approx [0.7, 1.4]$, eliminating the incentive to try a more talent-demanding task than the current-income maximizing task. Therefore, the optimal task m_0 is equal to one and the associated income y_0 is equal to one at $t = 0$. This outcome is constructed as a narrative benchmark in which the results mimic the uniform-talent case discussed in section 2.1. Nonetheless, the optimal talent threshold for staying in the initial trade $\check{\tau}_1$ is 1.07, greater than one. If the updated talent is moderately above $\bar{\tau} = 1$ at the beginning of $t = 1$, the worker moves to trade 1 since the upside income potential is higher when trying a task $\tilde{m} = 1$ in trade 1 (option 3) than trying a task within the talent-update task range $[\eta_l^0 \tilde{m}_0, \eta_h^0 \tilde{m}_0] = [0.5, 2]$ in trade 0 (option 1) given the larger sizes of talent updates in trade 1 (i.e., $\sigma_0 > \sigma_1$), or trying a task constrained to be outside the talent-update task range in trade 0 (option 2). Therefore, unlike in the two-period model, the trade-switch rate $F(\check{\tau}_1)$ is more than a 'failure' rate and includes workers drawn to another trade despite being able to earn more than the unconditional maximum income $\bar{y} = 1$ in the initial trade. The expected lifetime income $E[\sum_{t=0}^{2} y_t]$ is about 26% higher than 3.00 which would be obtained if the worker earned the unconditional maximum income $\bar{y} = 1$ in each period.

The cross-sectional income distribution at $t = 1$ exhibits a flat profile up to the 63rd percentile filled by switchers ($s_1 \neq s_0$), and then a rising profile subsequently filled by stayers ($s_1 = s_0$). The cross-sectional income distribution at $t = 2$ exhibits a similar pattern but with a smaller fraction of switchers ($s_2 \neq s_1$), 42 percent, and higher incomes of stayers ($s_2 = s_1$), some of whom have moved up the task ladder twice following two positive talent updates. The 90th-percentile income rises to 2.45 at $t = 2$. For a loose reference, Baker et. al. (1994) show that among the management employees entering a US service-sector firm in 1975, the 95th percentile salary as a multiple of the starting salary rose to about three by 1987. The income constraints (rows 17 to 29) do not change any of the outcomes since the optimal task delivers an income larger than 85 percent of the unconditional maximum income in each period.

Unconstrained Workers: $\tilde{\gamma}^0 = 3$ or $\tilde{\gamma}^0 = 5$ and $\mu = 0$

Columns 3 to 5 and Columns 6 to 8 in Tables 1 and 2 characterize the career tracks when $\tilde{\gamma}^0 = 3$ and $\tilde{\gamma}^0 = 5$, which imply that the fixed cost share of the gross output is 2/3 and 4/5, respectively. Consider the unconstrained worker (Columns 4 to 16). If $\mu = 0$, the interval $[\eta_l^0, \eta_h^0]$ stays constant as $\tilde{\gamma}^0$ rises, so the optimal first-period task $\tilde{m}_0$ remains at one. If $\mu = 0$ and $\rho = 1$, the optimal talent threshold for staying in the initial trade $\check{\tau}_1$ rises as $\tilde{\gamma}^0$ rises. As mentioned, given the larger sizes of talent updates in trade 1 (i.e., $\sigma_0 > \sigma_1$), the higher upside income potential in trade 1 draws workers whose $\tau_1(\tilde{m}_0, s_0)$ is moderately above one to trade 1, and this income advantage of trade 1 is amplified as both

$\tilde{\gamma}^0$ and $\tilde{\gamma}^1$ rise. If $\mu = 0$ and $\rho = 0$, $\check{\tau}_1$ instead falls as $\tilde{\gamma}^0$ rises. With $\rho = 0$, $\tilde{\gamma}^1$ stays at 1.5 while $\tilde{\gamma}^0$ rises, lowering the advantage of trade 1 instead. When $\tilde{\gamma}^0 = 5$, $\check{\tau}_1$ falls below one: Workers whose $\tau_1(\tilde{m}_0, s_0)$ are moderately below one tolerate a low current income (i.e., income below $\bar{y} = 1$) for the upside income potential at trade 0. Therefore, the fall of trade-switching disguises the low current-income stayers.

Unconstrained Workers: $\tilde{\gamma}^0 = 3$ or $\tilde{\gamma}^0 = 5$ and $\mu = 0.5$ or $\mu = 1$

If $\mu = 0.5$ or $\mu = 1$, as $\tilde{\gamma}^0$ rises, the interval $[\eta_l^0, \eta_h^0]$ becomes smaller creating the talent-task gaps for high-enough talent updates as in the two-period model. Consequently, the optimal first-period task $\tilde{m}_0$ rises, lowering the first-period income y_0 as in the two-period model. This implies that the lower task bound $\eta_l^0\tilde{m}_0^0$ (not shown in the table) rises as well, echoing Proposition 4. The upper task bound $\eta_h^0\tilde{m}_0^0$ (not shown in the table) falls as $\tilde{\gamma}^0$ rises except when $\tilde{\gamma}^0$ rises from 3 to 5 under $\mu = 0.5$ and $\rho = 1$. The optimal task $\tilde{m}_0$ rises more for a higher μ, which shrinks the interval $[\eta_l^0, \eta_h^0]$ by more as $\tilde{\gamma}^0$ rises. This implies that the lower task bound $\eta_l^0\tilde{m}_0^0$ rises as μ rises, echoing Proposition A2. On the other hand, the upper task bound $\eta_h^0\tilde{m}_0^0$ falls as μ rises, echoing Proposition A1.

If $\mu = 0.5$ or $\mu = 1$ and if $\rho = 1$, as $\tilde{\gamma}^0$ rises, the expected income from moving to trade 1 (option 3) rises along with the expected income from staying in trade 0. In particular, choosing a task without a task-range constraint in trade 1 (option 3) weakly dominates attempting a task outside the talent-update task range in trade 0 (option 2). Under the parameters chosen, the task range constraint is binding for option 2, so option 3 strictly dominates option 2. If $\mu = 0.5$, the income advantage of trade 1 due to the larger sizes of talent updates in trade 1 (i.e., $\sigma_0 > \sigma_1$) is amplified as $\tilde{\gamma}^0$ rises, raising $\check{\tau}_1$, as discussed for the case of $\mu = 0$. However, this effect is weaker than in the case of $\mu = 0$ because a higher μ narrows the range of the talent-update task range, which lowers the benefit of the larger sizes of talent updates. If $\mu = 1$, the effect nearly disappears and $\check{\tau}_1$ changes little as $\tilde{\gamma}^0$ rises.

If $\mu = 0.5$ or $\mu = 1$ and if $\rho = 0$, as $\tilde{\gamma}^0$ rises, the expected income from moving to trade 1 (option 3) stays constant while the upside income potential from staying in trade 0 rises, motivating the worker to stay in trade 0 and attempt a task outside the talent-update task range (option 2) if his updated talent is below a threshold, denoted by $\acute{\tau}_1$. Consequently, the probability of moving to trade 1 ($F_0(\check{\tau}_1)$) falls to zero while the probability of moving off the talent-update task range in trade 0 ($F_0(\acute{\tau}_1)$) is positive. The worker's current income y_1 from taking option 2 can be as low as 0.69 (when $\mu = 0.5$ and $\tilde{\gamma}^0 = 5$). The worker tolerates a low current income for the upside income potential at $t = 1$ as he does at $t = 0$. Therefore, the absence of trade-switchers disguises the low current-income stayers, which is qualitatively similar to the case of $\tilde{\gamma}^0 = 5$ under $\mu = 0$ and $\rho = 0$.

Unconstrained Workers: Income Changes

The unconstrained expected lifetime income $E[\sum_{t=0}^{2} y_t]$ grows substantially as $\tilde{\gamma}^0$ rises. The cross-sectional income distributions worsen as $\tilde{\gamma}^0$ rises. At $t = 1$, the 50-10 income ratio stays constant but the 90-50 ratio rises significantly, reflecting both positive talent updates delivering larger income gains and the negative talent updates leading to prolonged low incomes (options 2 or 3). At $t = 2$, the 50-10 income ratio rises but the 90-50 ratio rises much more. For a loose reference, the average CEO compensation of the top 350 US firms as a multiple of the average production worker compensation rose from 20 in 1965 to 278 in 2018 (Mishel and Wolfe 2019). If $\rho = 1$ or if $\mu = 0.5$ or $\mu = 1$, $\tilde{\gamma}^{s_1} = \tilde{\gamma}^0$ along any career paths (either $\rho = 1$ or $\rho = s_1 = 0$), as discussed above. As a consequence, the median income at $t = 2$ is (nearly) constant at 1.39 for $\tilde{\gamma}^0 = 3$ and 1.74 for $\tilde{\gamma}^0 = 5$, which reflects that the median income earners are workers who drew the final (expected) talent τ_2 of about 1.12 in trade s_1. If $\rho = 0$ and $\mu = 0$, workers who draw a low enough talent at the beginning of $t = 1$ move to trade 1 and face lower talent rewards (i.e., $\tilde{\gamma}^{s_1} = \tilde{\gamma}^1 < \tilde{\gamma}^0$) when $\tilde{\gamma}^0 = 3$ or $\tilde{\gamma}^0 = 5$, so their income growth is dampened and the median income at $t = 2$ is lower at 1.26 for $\tilde{\gamma}^0 = 3$ and 1.32 for $\tilde{\gamma}^0 = 5$.

One half or more of the workers earning about the median income at $t = 2$ are workers who moved to trade 1 (option 3) or moved off the talent-update task range in trade 0 (option 2) at $t = 1$, and the rest are workers who stayed within the talent-update task range in trade 0 (option 1) at $t = 1$. Thus, many of the final-period median income earners have gone through a prolonged period of low income, and the life-time income of the final-period medium income earner can fall as $\tilde{\gamma}^0$ rises. For example, if $\mu = 1$ and $\rho = 0$, the minimum life-time income of the final-period medium income earner is 3.18 ($y_0 = 1$, $y_1 = 1$, $y_2 = 1.18$) when $\tilde{\gamma}^0 = 1.5$. The life-time income of the final-period medium income earners who took option 2 at $t = 1$ is 3.26 ($y_0 = 0.87$, $y_1 = 1$, $y_2 = 1.39$) when $\tilde{\gamma}^0 = 3$, and 2.74 ($y_0 = 0.17$, $y_1 = 0.83$, $y_2 = 1.74$) when $\tilde{\gamma}^0 = 5$. Therefore, the rise of the final-period median income disguises a possibly large income loss along the career paths leading to the median income.

Constrained Workers: $\tilde{\gamma}^0 = 3$ or $\tilde{\gamma}^0 = 5$

Now consider constrained workers (rows 17 to 29). Under any values of μ and ρ considered, the income constraint of $\omega = 0.85$ is not binding when $\tilde{\gamma}^0 = 3$, so the outcome is identical to the unconstrained outcome. As mentioned, this is a narrative construction. When $\tilde{\gamma}^0 = 5$ and $\mu = 0$, the optimal first-period income y_0 is one, so the first-period task $\tilde{m}_0$ remains at one. When $\rho = 0$ in addition, the optimal second-period income y_1 of a worker who receives a talent update τ_1 above the threshold $\check{\tau}_1 = 0.96$ but below one, is less than one as was discussed. Nonetheless, income constraints are not binding as the sum of the optimal incomes over the first two periods remains above the constraint threshold: $y_0 + y_1 > 2\omega = 1.7$.

When $\tilde{\gamma}^0 = 5$ and $\mu = 0.5$ or $\mu = 1$, the income constraint in the first-period is binding and pulls down the first-period task $\tilde{m}_0$ in order to generate the first-period income $y_0 \geq 0.85$. In the table, $y_0 = 0.87$, which is the lowest value of $y_0 \geq 0.85$ when $\tilde{\gamma}^0 = 5$ and $\tau_0 = \bar{\tau} = 1$ due to the grid-based computation (151 possible values of τ or $\tilde{m}$ with the ratio of adjacent values equal to 1.014). This generates a rising and then falling level of the first-period task $\tilde{m}_0$ as $\tilde{\gamma}^0$ rises from 1.5 to 3 and then from 3 to 5, echoing Proposition 2. The constrained career paths lead to a smaller growth of the expected lifetime income and of some segments of the cross-sectional income profiles, as $\tilde{\gamma}^0$ rises. If $\mu = 0.5$ and $\rho = 0$, the optimal second-period income y_1 of a worker who stays off the talent-update task range in trade 0 (option 2) following a talent update τ_1 below the threshold $\acute{\tau}_1 = 1.03$, falls below $\omega = 0.85$. Unlike the case of $\mu = 0$ and $\rho = 0$ discussed above, the income constraint is binding in the second period as well as in the first period, forcing workers with τ_1 below $\check{\tau}_1 = 0.97$ to move to trade 1 (option 3). Note that the constrained career paths lead to a fall of the final-period median income below that under $\tilde{\gamma}^0 = 3$ (from 1.39 to 1.32). Therefore, the final-period median income can first rise and then fall as $\tilde{\gamma}^0$ rises from 1.5 to 3 and then from 3 to 5.

References for Appendix B

Baker, G., Gibbs, M., and Holmstrom, B. (1994), "The Wage Policy of a Firm," *Quarterly Journal of Economics* 109:921-955.

Mishel, L., and Wolfe, J. (2019), *CEO Compensation Has Grown 940% since 1978*, Economic Policy Institute Report (epi.org/171191).

Table 1: Numerical Exercise under $\rho = 1$ and $\nu = 0.5$

$\tilde{\gamma}^0$	1.5	3.0	3.0	3.0	5.0	5.0	5.0
μ	Any	0.0	0.5	1.0	0.0	0.5	1.0
ρ	Any	1.0	1.0	1.0	1.0	1.0	1.0
$\tilde{m}_0\|_{\text{optimal}}$	1.00	1.00	1.06	1.16	1.00	1.12	1.21
$y_0\|_{\text{optimal}}$	1.00	1.00	0.99	0.91	1.00	0.83	0.31
$\check{\tau}_1\|_{\text{optimal}}$	1.07	1.10	1.10	1.07	1.15	1.13	1.07
$F_0(\check{\tau}_1)\|_{\text{optimal}}$	0.63	0.65	0.65	0.63	0.68	0.67	0.63
$\acute{\tau}_1\|_{\text{optimal}}$	-	-	-	-	-	-	-
$F_0(\acute{\tau}_1)\|_{\text{optimal}}$	-	-	-	-	-	-	-
$E[\sum_{t=0}^{2} y_t]\|_{\text{optimal}}$	3.79	5.58	5.52	5.14	12.70	12.00	7.78
$y_1(10\text{th})\|_{\text{optimal}}$	1.00	1.00	0.99	0.97	1.00	0.87	0.77
$y_1(50\text{th})\|_{\text{optimal}}$	1.00	1.00	0.99	0.97	1.00	0.87	0.77
$y_1(90\text{th})\|_{\text{optimal}}$	2.16	4.66	4.65	4.28	13.00	12.97	9.32
$y_2(10\text{th})\|_{\text{optimal}}$	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$y_2(50\text{th})\|_{\text{optimal}}$	1.18	1.39	1.39	1.39	1.74	1.74	1.74
$y_2(90\text{th})\|_{\text{optimal}}$	2.45	6.23	6.00	5.33	21.11	19.70	11.55
$\tilde{m}_0\|_{\omega=0.85}$	1.00	1.00	1.06	1.16	1.00	1.10	1.10
$y_0\|_{\omega=0.85}$	1.00	1.00	0.99	0.91	1.00	0.87	0.87
$\check{\tau}_1\|_{\omega=0.85}$	1.07	1.10	1.10	1.07	1.15	1.13	1.07
$F_0(\check{\tau}_1)\|_{\omega=0.85}$	0.63	0.65	0.65	0.63	0.68	0.67	0.63
$\acute{\tau}_1\|_{\omega=0.85}$	-	-	-	-	-	-	-
$F_0(\acute{\tau}_1)\|_{\omega=0.85}$	-	-	-	-	-	-	-
$E[\sum_{t=0}^{2} y_t]\|_{\omega=0.85}$	3.79	5.58	5.52	5.14	12.70	11.99	7.40
$y_1(10\text{th})\|_{\omega=0.85}$	1.00	1.00	0.99	0.97	1.00	0.87	0.87
$y_1(50\text{th})\|_{\omega=0.85}$	1.00	1.00	0.99	0.97	1.00	0.87	0.87
$y_1(90\text{th})\|_{\omega=0.85}$	2.16	4.66	4.65	4.28	13.00	12.90	7.32
$y_2(10\text{th})\|_{\omega=0.85}$	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$y_2(50\text{th})\|_{\omega=0.85}$	1.18	1.39	1.39	1.39	1.74	1.74	1.74
$y_2(90\text{th})\|_{\omega=0.85}$	2.45	6.23	6.00	5.33	21.11	19.70	10.04

Table 2: Numerical Exercise under $\rho = 0$ and $\nu = 0.5$

$\tilde{\gamma}^0$	1.5	3.0	3.0	3.0	5.0	5.0	5.0
μ	Any	0.0	0.5	1.0	0.0	0.5	1.0
ρ	Any	0.0	0.0	0.0	0.0	0.0	0.0
$\tilde{m}_0\|_{\text{optimal}}$	1.00	1.00	1.15	1.20	1.00	1.16	1.23
$y_0\|_{\text{optimal}}$	1.00	1.00	0.93	0.87	1.00	0.63	0.17
$\check{\tau}_1\|_{\text{optimal}}$	1.07	1.00	-	-	0.96	-	-
$F_0(\check{\tau}_1)\|_{\text{optimal}}$	0.63	0.58	-	-	0.55	-	-
$\acute{\tau}_1\|_{\text{optimal}}$	-	-	1.03	1.07	-	1.03	1.07
$F_0(\acute{\tau}_1)\|_{\text{optimal}}$	-	-	0.60	0.63	-	0.60	0.63
$E[\sum_{t=0}^{2} y_t]\|_{\text{optimal}}$	3.79	5.20	5.23	5.10	10.96	10.91	7.77
$y_1(10\text{th})\|_{\text{optimal}}$	1.00	1.00	0.83	1.00	1.00	0.69	0.83
$y_1(50\text{th})\|_{\text{optimal}}$	1.00	1.00	0.83	1.00	1.00	0.69	0.83
$y_1(90\text{th})\|_{\text{optimal}}$	2.16	4.66	4.66	4.36	13.00	13.00	9.61
$y_2(10\text{th})\|_{\text{optimal}}$	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$y_2(50\text{th})\|_{\text{optimal}}$	1.18	1.26	1.39	1.39	1.32	1.74	1.74
$y_2(90\text{th})\|_{\text{optimal}}$	2.45	4.66	4.79	4.96	13.00	13.00	11.33
$\tilde{m}_0\|_{\omega=0.85}$	1.00	1.00	1.15	1.20	1.00	1.10	1.10
$y_0\|_{\omega=0.85}$	1.00	1.00	0.93	0.87	1.00	0.87	0.87
$\check{\tau}_1\|_{\omega=0.85}$	1.07	1.00	-	-	0.96	0.97	-
$F_0(\check{\tau}_1)\|_{\omega=0.85}$	0.63	0.58	-	-	0.55	0.56	-
$\acute{\tau}_1\|_{\omega=0.85}$	-	-	1.03	1.07	-	-	1.06
$F_0(\acute{\tau}_1)\|_{\omega=0.85}$	-	-	0.60	0.63	-	-	0.62
$E[\sum_{t=0}^{2} y_t]\|_{\omega=0.85}$	3.79	5.20	5.23	5.10	10.96	10.52	7.23
$y_1(10\text{th})\|_{\omega=0.85}$	1.00	1.00	0.83	1.00	1.00	1.00	1.00
$y_1(50\text{th})\|_{\omega=0.85}$	1.00	1.00	0.83	1.00	1.00	1.00	1.00
$y_1(90\text{th})\|_{\omega=0.85}$	2.16	4.66	4.66	4.36	13.00	12.90	7.32
$y_2(10\text{th})\|_{\omega=0.85}$	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$y_2(50\text{th})\|_{\omega=0.85}$	1.18	1.26	1.39	1.39	1.32	1.32	1.74
$y_2(90\text{th})\|_{\omega=0.85}$	2.45	4.66	4.79	4.96	13.00	13.00	9.02