Back to the St. Petersburg Paradox?

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Abstract:

Conventional parameterizations of cumulative prospect theory do not explain the St. Petersburg paradox. To do so, the power coefficient of an individual's utility function must be lower than the power coefficient of an individual's probability weighting function.

Abstrakt:

Konvenční parametrizace kumulativní prospektové teorie nevysvětlují Petrohradský paradox. K tomu je zapotřebí, aby mocnitel jedincovy užitkové funkce byl nižší než mocnitel jeho pravděpodobnostní váhové funkce.

Keywords: EUT, cumulative prospect theory, St. Petersburg paradox, power utility, probability weighting

JEL Classification codes: C91, D81

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Back to the St. Petersburg paradox?

The St. Petersburg paradox (Bernoulli, 1738) refers to a lottery L that delivers an outcome 2^n with probability $2^{-n}, n \in \mathbb{N}$. The maximum price that an individual is willing to pay for L is finite and typically low. However, L has an infinite expected value. Thus, the St. Petersburg paradox is generally taken as evidence against expected value and in favor of expected utility theory (EUT). Samuelson (1977) offers an extensive survey of the St. Petersburg paradox.

Arguably the dominant descriptive decision theory today is cumulative prospect theory or CPT (Tversky and Kahneman, 1992). CPT accommodates a large amount of experimental data including robust violations of EUT such as the Allais paradox (Allais, 1953). According to CPT an individual utility of the lottery L involved in the St. Petersburg paradox is given by formula (1), where $u : \mathbb{R}_+ \to \mathbb{R}_+$ is an individual's utility function for gains and $w : [0,1] \to [0,1]$ is an individual's probability weighting function for gains.

$$u(L) = \sum_{n=1}^{+\infty} u(2^n)\left[w(2^{1-n}) - w(2^{-n})\right] \qquad (1)$$

Following Tversky and Kahneman (1992), the majority of studies adopt a power utility function $u(x) = x^\alpha$ and an S-shaped probability weighting function $w(p) = p^\gamma / \left(p^\gamma + (1-p)^\gamma\right)^{1/\gamma}$ first proposed by Quiggin (1982). Since the St. Petersburg paradox lottery L involves very small probabilities, Quiggin's function $w(p)$ may be accurately approximated as $w(p) \approx p^\gamma$ because the denominator of Quiggin's function $w(p)$ converges to unity for tiny probabilities p. Then, equation (1) simplifies into formula (2).

$$u(L) \approx (2^\gamma - 1) \sum_{n=1}^{+\infty} 2^{(\alpha-\gamma)n} \qquad (2)$$

It follows from (2) that according to CPT an individual obtains a bounded utility from lottery L only when $\alpha < \gamma$ *i.e.* when the sum on the right hand side of (2) is convergent. Thus, CPT explains the St. Petersburg paradox only when the power coefficient of an individual's utility function is lower than the power coefficient of an individual's probability weighting function. Intuitively, an individual's utility function must not simply be concave but it must be concave relative to an individual's probability weighting function to avoid the St. Petersburg paradox.

Table 1 presents typical values of power coefficients α and γ that were obtained from the best parametric fitting to the experimental data in well-known recent studies. Some studies (*e.g.* Tversky and Fox, 1995) adopted a probability weighting function $w(p) = \delta \cdot p^{\gamma} / \left(\delta \cdot p^{\gamma} + (1-p)^{\gamma}\right)$, first used by Goldstein and Einhorn (1987). For small probabilities a Goldstein-Einhorn function $w(p)$ can be approximated as $w(p) \approx \delta \cdot p^{\gamma}$. An individual then still obtains a bounded utility from lottery L only when $\alpha < \gamma$. The best fitting estimates of a power coefficient γ for a Goldstein-Einhorn function $w(p)$ are presented in parentheses in the third column of table 1 (for those studies where applicable).

In all studies from table 1 except for Camerer and Ho (1994) and Wu and Gonzalez (1996) the estimated best fitting CPT parameters are $\alpha > \gamma$, which implies a divergent sum on the right hand side of equation (2). Thus, conventional parameterizations of CPT predict that an individual is willing to pay up to infinity for the St. Petersburg lottery L. This paradoxical result occurs because a conventional inverse S-shaped probability weighting function overweights small probabilities too much for a mildly concave utility function to offset this effect.

Apparently, the parameterization of CPT that accommodates best the available experimental evidence does not explain the oldest and the most famous paradox in decision theory—the St. Petersburg paradox. To accommodate the St. Petersburg paradox CPT must be estimated together with a restriction $\alpha < \gamma$ on its parameters. However, it is not obvious if a restricted version of CPT remains descriptively superior to other decision theories.

Experimental study	Power of utility function (alpha)	Power of probability weighting function (gamma)
Kahneman and Tversky (1992)	0.88	0.61
Camerer and Ho (1994)	0.37	0.56
Tversky and Fox (1995)	0.88	(0.69)
Wu and Gonzalez (1996)	0.52	0.71 (0.68)
Abdellaoui (2000)	0.89	0.60 (0.60)
Bleichrodt and Pinto (2000)	0.77	0.67 (0.55)
Kilka and Weber (2001)	0.76 – 1.00	(0.30 – 0.51)
Abdellaoui et al. (2003)	0.91	(0.76)

Table 1 Parameterization of CPT that accommodates best the experimental data in well-known recent studies

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