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Electing the pope: Elections by repeated ballots

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Abstract

A finite group of voters must elect the pope from a finite set of candidates. They repeatedly cast ballots (possibly for ever) until one candidate attains at least Q votes. A candidate is *electable*—if enough voters prefer him to a continuous disagreement—as well as *stable*—if no other candidate is preferred to him by a sufficient number of voters. We provide a necessary and sufficient condition for the existence of a candidate that is both electable and stable. When there are three candidates and voters are willing to compromise somewhat, the condition requires choice by two-thirds supermajority, which coincides with the procedure that the Catholic Church has used to appoint the pope for almost a millennium.

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Keywords: repeated ballots, conclave, pope, electable, stable, supermajority

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1 Introduction

The election of the pope by the College of Cardinals is perhaps the most structured and long lasting example of an electoral institution. The Vatican conclave follows precise rules developed and refined over centuries whereby *repeated ballots* take place for as long as it takes (possibly forever) until one of the candidates receives 2/3 of the votes.

The combination of a 2/3 supermajority requirement with repeated ballots in the Vatican conclave appears to be geared to attain a good compromise between the desire to elect a stable pope, and the need to avoid gridlock. The cardinals wish to elected a stable pope. That is, the elected pope ought to be sure to win if he were to face a vote against any other candidate. But it has long been known[1] that with more than two candidates a simple majority may easily elect an unstable candidate. So the goal of stability advises a supermajority, the stronger the better. On the other hand, the cardinals need to elect someone, and the chances of gridlock are greater the stronger the majority required is. So, to help electability, a minimal majority, not a 2/3 supermajority, would be advisable. The Catholic Church seems to have settled at the 2/3 rule as the 'right' compromise between electability and stability.

However, reaching an outcome under a 2/3 supermajority is difficult. Repeated ballots are a tool to facilitate compromise. In unstructured settings, when a vote is inconclusive, it is natural that another vote takes place at a later date, and so on, again and again if necessary. So the institutionalisation of repeated ballots appears as a natural response to gridlock. With repeated ballots, voters must take into consideration the relative value of a gridlock (staying locked in the Vatican voting forever without an outcome does not seem too attractive), vis-à-vis the election of each of the candidates. For each cardinal, some candidates are surely better than gridlock, they are acceptable, and others are not. Thus voters might eventually compromise and vote for one such acceptable candidate, even one ranking rather low in the preference profile. When an election requires Q votes, the existence of an electable candidate, one acceptable for at least Q voters, is a necessary condition to assure that voting will not go on forever.

With the motivating example of the Vatican conclave in mind, we propose a general model of elections by repeated ballots to explore the conditions under which electability and stability, each in turn, and both at the same time, prevail. We consider n voters that must choose one candidate from a finite set of k candidates. We assume that repeated ballots take place until a candidate obtains at least $Q \geq \frac{n+1}{2}$ votes. Each individual has a strict preference order over the set of candidates, and can compare each candidate to the prospect of gridlock. We show that for a given electorate, increasing Q expands the set of stable candidates and shrinks the set of electable candidates. We provide the

[1] By Pliny the Younger, Ramon Llull, Condorcet, Borda and Arrow, among others (McLean and Urken, 1995).

necessary and sufficient condition for a non-empty set of electable candidates, and the necessary and sufficient condition for a non-empty set of stable candidates. Satisfying both conditions is not sufficient for the existence of a candidate that is both electable and stable. Our main contribution is to provide a necessary and sufficient conditions that assures the existence of a candidate that is both electable and stable. When there are three candidates and when voters are willing to compromise 'somewhat', assumptions we argue oftentimes fit the reality of papal elections, the unique voting rule that satisfies the condition is precisely the 2/3 supermajority.

Repeated ballots: Although perhaps not as structured as in the case of the pope election, repeated ballots—balloting until a candidate obtains a predefined number of votes without eliminating candidates between ballots as in runoff elections—are a common procedure to (s)elect top-ranking appointees in all kinds of organizations. They are used to appoint of heads of state, US presidential candidates, and orchestra conductors.

The Third Lateran Council in 1179 established the 2/3 supermajority required to elect a pope. The rules of balloting *cum clave*, with a key, to elect the new pope were laid out by Pope Gregory X in 1274. Pope Gregory XV in 1621 restricted ballots to include a single name and prohibited a cardinal from voting for himself. Since then the rules remain largely unchanged (Baumgartner, 2003, p. 33, 40, 145, 147).[2] The rules, however, do not clarify the meaning of the 2/3 supermajority when the number of cardinals is not divisible by three. One exception is Pope John Paul II, who rules that 'two thirds of the votes are required' and in the case of indivisibility 'one additional vote is required' (John Paul II, 1996, §62). The subsequent 2005 conclave that elected Pope Benedict XVI required 77 votes from 115 cardinals present (Allen, 2005, p. 112), which is 2/3 rounded up. Another exception similar to the previous one is Pope John XXIII. Seven of the eleven conclaves that took place since the beginning of the 20th century had the number of cardinals not divisible by three. In each case the number of votes required was 2/3 of the number of cardinals rounded up (Baumgartner, 2003; Walsh, 2003; Allen, 2005; O'Connell, 2021; O'Connell and Piqué, 2025).[3]

Repeated ballots are used to elect the Italian president. The joint session of the Italian Parliament votes until a candidate receives the required majority, which is 2/3

[2]Several minor changes include the switch to $2/3+1$ supermajority (and a removal of the prohibition of voting for self) (Pius XII, 1945, §68), reversal of this change (John XXIII, 1962, §15), reinstatement (Paul VI, 1975, §65), and a final reversal (John Paul II, 1996, §62). Pope John Paul II decreed that after 30 ballots a simple majority suffices for the election and restricted the choice to the top two vote-getters (John Paul II, 1996, §75). Pope Benedict XVI changed back to the 2/3 supermajority, but upheld the top two vote-getters after the 30th ballot provision (Benedict XVI, 2013, §75).

[3]In addition to repeated ballots, a method typically called 'scrutiny', the other methods to elect a pope are by 'acclamation' (all cardinals verbally assent to a candidate) and by 'compromise' (delegating the choice to an ad hoc committee). The last time a pope has been elected by acclamation is in 17th century and by compromise is in 13th century (Piazzoni, 2017). Pope Gregory XI made scrutiny the usual method in 1621 (Walsh, 2003, p. 126) and Pope John Paul II banned the other two methods in 1996 (John Paul II, 1996, §62).

supermajority on the first three ballots and simple majority thereafter. The process can be lengthy: 16 rounds of voting were necessary in 1992, and 23 in 1971. In 2013 the appointment of a successor to Italian President Giorgio Napolitano was due. Negotiations dragged on for weeks and no candidate attained the required number of votes in the first five ballots. The election was unsolvable until the incumbent president agreed to run for another term and was elected on the sixth ballot.[4]

The US president might be elected via repeated balloting. When no candidate receives a simple majority of the votes in the Electoral College, a contingent election—repeated balloting in the US House of Representatives until a candidate receives simple majority, with the choice restricted to three candidates who receive the most electoral college votes—is used to elect the president. In 1825, John Quincy Adams was elected US president in a contingent election. In 1837, Richard Mentor Johnson was elected US vice president in a contingent election. Both in a single ballot. In 1801, Thomas Jefferson was elected US president, this time the election took six days of debates and 36 ballots.[5]

The two main political parties in the US select their presidential candidates at National Conventions. When no candidate receives a simple majority of delegates' votes on the first ballot, the convention becomes brokered and repeated balloting follows until a candidate secures support from a simple majority of delegates. At least ten conventions of each party were brokered, the last Democratic one in 1952 and the last Republican in 1948. The 1880 Republican Convention needed 36 ballots to elect James A. Garfield, a dark horse, who up to the 33rd ballot received at most two votes, far away from the required 379. The 1924 Democratic Convention took a record 103 ballots to nominate presidential candidate John W. Davis, a compromise candidate following a deadlock between William Gibbs McAdoo and Al Smith. From 1832 to 1932 the Democratic Party required a 2/3 supermajority, as in the papal conclave.[6]

Repeated ballots are even used to appoint orchestra conductors. The 2015 choice of the chief conductor of the Berlin Philharmonic to succeed Sir Simon Rattle has been likened to the 'papal conclave of the music world'. The process has been shrouded in

[4]The required majorities are specified in Article 83 of the Italian Constitution. Lengthy elections: Guardian: With Parliament in Deadlock, Italy Seeks a President, April 18, 2013. The 2013 election: New York Times: Italy's President Is Granted New Term in Last-Ditch Effort to Break Deadlock, April 21, 2013.

[5]The Twelfth Amendment of the US Constitution ratified in 1804 provides for contingent elections. Contingent election of the vice president happens in the US Senate. The choice is restricted to two candidates and is by simple majority. For the 1825 and 1837 contingent elections see Neale (2020). For the 1801 contingent election see Ferling (2004, ch. 12). This was the last presidential election held before the Twelfth Amendment, which mainly changed voting in the electoral college, and left contingent elections almost unchanged.

[6]The Call For the 2024 Democratic National Convention in part IX.C.7.f decrees that 'balloting will continue until a nominee is selected'. The Rules of the Republican Party for the 2024 convention in rule 40(e) decree that the 'chairman of the convention [...] shall repeat the calling of the roll until a candidate shall have received a majority'. Regarding brokered conventions and 2/3 supermajority see David, Goldman, and Bain (1960, ch. 9, 17). For the 1880 convention see Ackerman (2003, ch. 5, 6). On the 1924 convention see New Yorker: Conventional Wisdom, March 21, 2016.

secrecy: the orchestra members met at a secret Berlin location and were not allowed to carry their phones. After 11 hours and several rounds of voting, they abandoned the effort and the subsequent conclave elected Kirill Petrenko a month later.[7]

Related literature: The political science and political economics literature analyzing papal elections is remarkably scarce. Colomer and McLean (1998) review the history of papal conclaves under the eyes of modern social choice theory, emphasizing the stability properties of the 2/3 majority rule. Mackenzie (2020) is concerned with the potential problem of cardinals voting for themselves and advocates a return to the rule dictated in 1945 by Pope Pius XII that required $2/3 + 1$. Kwiek (2014) proposes a non-cooperative game to model the conclave. He assumes that there are only two candidates, and that each voter has an upper limit on the delay he is ready to endure, which is a commonly known. Consequently the equilibrium outcome is an immediate election where the candidate preferred by the pivotal voter (in terms of the order induced by the endurance limits) prevails. Hunt (2015) studies betting on papal elections.

Maltzman, Schwartzberg, and Sigelman (2006) discuss the 2005 conclave that elected Ratzinger as Pope Benedict XVI. They conjecture that John Paul II had changed the rules of the conclave out of concerns for stability, after a workshop on democracy organized by the Pontifical Academy of Social Sciences, of which Kenneth J. Arrow was a member. These changes—election restricted to the top two candidates and simple-majority after the 30th ballot—were in effect only in 2005, because Pope Benedict XVI himself ordered to return to 2/3 for all ballots.

In the ensuing conclave, Bergoglio, who had been the only serious contender against Ratzinger, was elected as Pope Francis. Kóczy and Sziklai (2015) propose an analysis of the 2013 conclave (where 115 cardinals participated, and 77 votes were necessary) through the measurement of power indices à la Shapley and Shubik (1954). First, they identify each cardinal as a pair (x, y), where x is the distance from his birthplace to Rome and y is an indicator of his ideology in the conservative-liberal axis. Then they propose an algorithm that checks all minimal winning coalitions (rectangular subsets of 77 cardinals) and gives one point to coalition members that are pivotal (i.e., sit on the boundary). Bergoglio's score turns out to be quite high, he ranks third.

More generally, our work is a contribution to the literature concerned with the stability of collective decisions taken by voting, and the properties of supermajority requirements. As is well known, the prevalence of Condorcet cycles of intransitive majority preferences—when a majority prefers A to B, a majority prefers B to C, and a majority prefers C to A—are the cornerstone of Arrow's foundational theorem. Ruling out such cycles requires a restricted domain of preferences (the simplest one are single-peaked preferences over one-dimensional candidates), or a stronger majority, often a very strong

[7] Guardian (Online): Berlin Philharmonic Deadlocked over Simon Rattle's Successor, May 11, 2015 and Guardian: Kirill Petrenko to Succeed Simon Rattle at the Berlin Philharmonic, June 28, 2015.

supermajority. Black (1948) was the first to study the stability properties of supermajority voting rules. Further developments exploring stability of voting rules formalize voting as a cooperative game and examine conditions for the (non)existence of the core. Moulin (1988) and Austen-Smith and Banks (2000) provide comprehensive treatments.[8] For n dimensional sets of candidates Caplin and Nalebuff (1988, 1991) show that a stable candidate exists under 64% supermajority provided that voters have Euclidean preferences with sufficiently similar peaks, thus providing support for a quota rule remarkably close to the 2/3 requirement. Other papers in the literature point out the benefits of supermajority requirements as a tool to resolve trade-offs between commitment and flexibility (Aghion and Bolton, 2003; Aghion, Alesina, and Trebbi, 2004; Barbera and Jackson, 2004; Dal Bó, 2006; Messner and Polborn, 2004), or between different costs of collective action (Buchanan and Tullock, 1962, ch. 6). We consider unrestricted preference over finite sets of candidates, and focus on the link between stability and electability, a novel property of candidates induced by the interaction of preferences over candidates and gridlock under repeated ballots.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 presents the results. Appendix A contains all the proofs.

2 Model

We study the existence of feasible candidates that may arise victorious when a set of voters chooses by repeatedly casting ballots. Each voter has preferences over candidates but only casts votes for those candidates she prefers to a continuous disagreement. A candidate is feasible if he is *electable*—if enough voters prefer him to a continuous disagreement—as well as *stable*—if no other candidate is preferred to him by a sufficient number of voters.

Formally, $n \geq 2$ voters choose from among $k \geq 2$ candidates. Let $N = \{1, \ldots, n\}$ be the set of voters and $K = \{1, \ldots, k\}$ be the set of candidates. The choice of a candidate requires $Q \in \mathbb{N}$ or more votes, where $Q \geq \frac{n+1}{2}$ and $Q \leq n$. The voters either elect a candidate or remain in a *continuous disagreement*, an outcome we denote by d.

Each voter $i \in N$ has a complete, transitive and strict preference relation $\succ_i$ over K. Let P be the set of all such relations. Moreover, voters' preferences rank each candidate relative to the continuous disagreement: for any $i \in N$, any $\succ_i \in P$ and any $c \in K$, either $c \succ_i d$ or $d \succ_i c$. When $c \succ_i d$, we say that candidate c is *acceptable* for voter i given $\succ_i$ (with $\succ_i$ omitted when no confusion arises) and is *unacceptable* if $d \succ_i c$. We model i's willingness to compromise by a parameter $a_i \in \{0, \ldots, k\}$, which is the number of candidates voter i finds acceptable: $a_i = |\{c \in K : c \succ_i d\}|$ for any $\succ_i \in P$. The remaining $k - a_i$ candidates are unacceptable.[9] An *electorate* $\mathbf{v} = (\succ_i)_{i \in N} \in \mathbf{V} = P^n$

[8]See also our discussion following Proposition 1.

[9]An alternative modelling approach would be to endow voters with preferences over both $K \cup \{d\}$.

is a profile of voters' preferences over the candidates, where $\mathbf{V}$ is the set of all possible electorates. An *election* is $(\mathbf{v}, Q)$.

Candidate $c \in K$ is *electable* in election $(\mathbf{v}, Q)$ if $|\{i \in N : c \succ_i d\}| \geq Q$. That is, a candidate is electable if he is acceptable for Q or more voters. Conversely, a candidate is not electable if he is unacceptable for $n - Q + 1$ or more voters. A candidate may win an appointment by repeated ballots only if he is electable. If he is not, $n - Q + 1$ or more voters prefer continuous disagreement over casting a vote in his favour so that he receives at most $Q - 1$ votes. Let $\mathcal{E}(\mathbf{v}, Q)$ be the set of electable candidates in election $(\mathbf{v}, Q)$.

Candidate $c \in K$ is *stable* in election $(\mathbf{v}, Q)$ if $|\{i \in N : c' \succ_i c\}| < Q$ for any other candidate $c' \in K \backslash \{c\}$. That is, a candidate is stable if no other candidate is preferred by Q or more voters. Conversely, a candidate that is not stable may be challenged by a counter-candidate backed up by a coalition of Q or more voters. We insist on candidates being stable because candidates that are not may fail to be seen as legitimate leaders. When elected, a candidate that is not stable may spend most of his tenure trying to maintain authority and grip on power rather than governing. Tenures of unstable candidates thus may be tumultuous, futile or brief. In either case, unstable candidates are not desirable. Let $\mathcal{S}(\mathbf{v}, Q)$ be the set of stable candidates in election $(\mathbf{v}, Q)$.

3 Results

Our main research question is whether it is possible to guarantee the existence of stable and electable candidates for arbitrary profile of voters' preferences, that is, for any electorate $\mathbf{v}$. Example 1 show that stability and electability of a candidate are distinct properties.

Example 1. Consider an election with three voters and three candidates. The voters' preferences $\mathbf{v} = (\succ_1, \succ_2, \succ_3)$ over the candidates are in the following table.

1	2	3
2	3	1
3	1	2

The first column shows the preferences of the first voter with more preferred candidates placed higher, so that her ranking is $1 \succ_1 2 \succ_1 3$, and similarly for the remaining voters. The candidates shown with grey background are unacceptable, so that $(a_i)_{i \in N} = (2, 2, 2)$.

When $Q = 2$, then two voters prefer candidate 1 to candidate 2, two voters prefer candidate 2 to candidate 3, and two voters prefer candidate 3 to candidate 1. In particular, none of the candidates are stable, $\mathcal{S}(\mathbf{v}, Q) = \varnothing$. At the same time, all candidates

Because any preference profile in this alternative model maps into a preference profile in our model with a certain number of acceptable candidates, these two models are equivalent. Our model simplifies the exposition because most of our results are claims that hold for any voters' preference keeping their willingness to compromise fixed.

are electable, $\mathcal{E}(\mathbf{v}, Q) = \{1, 2, 3\}$, because each candidate is acceptable for two voters. However, changing a_i to $a_i = 1$ for all voters would make all voters unelectable.

When $Q = 3$, then for any pair of distinct candidates c and c' at most two voters prefer candidate c to candidate c'. Therefore, all candidates are stable, $\mathcal{S}(\mathbf{v}, Q) = \{1, 2, 3\}$. At the same time, none of the candidates are electable, $\mathcal{E}(\mathbf{v}, Q) = \varnothing$, because each of the candidates is unacceptable for one voter. However, changing a_i to $a_i = 3$ for all voters would make all voters electable.

This example shows that a candidate can be both electable and stable, can be neither, or posses only one of these properties. Note also that the increase in Q expands the set of stable candidates and shrinks the set of electable candidates. This turns out to be a general feature that holds beyond Example 1.

Lemma 1. *Suppose $Q < Q'$. For any $\mathbf{v} \in \mathbf{V}$, $\mathcal{S}(\mathbf{v}, Q) \subseteq \mathcal{S}(\mathbf{v}, Q')$ and $\mathcal{E}(\mathbf{v}, Q') \subseteq \mathcal{E}(\mathbf{v}, Q)$.*

The lemma is a direct consequence of the definitions of stability and electability. When a candidate is stable in election $(\mathbf{v}, Q)$, no other candidate is preferred by Q or more votes, and hence he is stable also in election $(\mathbf{v}, Q')$ when $Q' > Q$. When a candidate is electable in election $(\mathbf{v}, Q')$, he is acceptable for Q' or more voters, and hence he is electable also in election $(\mathbf{v}, Q)$ when $Q < Q'$.

Lemma 1 shows that for any electorate, increasing Q weakly expands the set of stable candidates and weakly shrinks the set of electable candidates. Conversely, decreasing Q weakly shrinks the set of stable candidates and weakly expands the set of electable candidates. In other words, when choosing Q that ensures the existence of stable and electable candidates, larger Qs favour stability but compromise electability, while lower Qs favour electability but compromise stability.

The following two propositions provide a sufficient and necessary conditions such that the set of stable candidates is non-empty for any electorate (Proposition 1) and such that the set of electable candidates is non-empty for any electorate (Proposition 2). In line with the discussion of Lemma 1 in the previous paragraph, ensuring stability requires Q above a lower bound, while ensuring electability requires Q below an upper bound.

Proposition 1. *$\mathcal{S}(\mathbf{v}, Q) \neq \varnothing$ for all $\mathbf{v} \in \mathbf{V}$ if and only if $Q > n\frac{k-1}{k}$.*

The proposition implies that a stable candidate exists for any electorate $\mathbf{v}$ if and only if Q strictly exceeds a lower bound equal to $n\frac{k-1}{k}$. To understand the lower bound, suppose we wish to construct an electorate $\mathbf{v}$ such that $\mathcal{S}(\mathbf{v}, Q) = \varnothing$. The construction allocates preference fragments $1 \succ_i 2$, $2 \succ_i 3$, ..., $k-1 \succ_i k$ and $k \succ_i 1$ to the voters. Allocating $1 \succ_i 2$ to Q voters makes candidate 2 unstable. Making no candidate stable thus requires allocating each of the k preference fragments to Q votes, that is, allocating Qk preference fragments overall. The total number of preference fragments that can be allocated to the voters is $n(k-1)$ because no voter can be allocated all the k preference fragments, as that would make her preferences intransitive. Our construction is thus feasible when $Qk \leq n(k-1)$. Conversely, if an electorate $\mathbf{v}$ such that $\mathcal{S}(\mathbf{v},Q) = \varnothing$ exists, then it must have $m \leq k$ preference fragments $1 \succ_i 2$, $2 \succ_i 3$, ..., $m-1 \succ_i m$ and $m \succ_i 1$ each allocated to at least Q voters. Transitivity of voters' preferences requires $Qm \leq n(m-1)$. This condition, because $\frac{m-1}{m}$ increases with m, in turn implies $Qk \leq n(k-1)$.[10]

Proposition 2. *$\mathcal{E}(\mathbf{v},Q) \neq \varnothing$ for all $\mathbf{v} \in \mathbf{V}$ if and only if $Q < \frac{\sum_{i \in N} a_i}{k} + 1$.*

The proposition implies that an electable candidate exists for any electorate $\mathbf{v}$ if and only if Q is strictly lower than an upper bound equal to $\frac{\sum_{i \in N} a_i}{k} + 1$. To understand the upper bound, suppose we wish to construct an electorate $\mathbf{v}$ such that $\mathcal{E}(\mathbf{v},Q) = \varnothing$. The construction places candidates 1, 2, ..., k to the positions in voters' preferences where candidates are unacceptable. Each voter has $k - a_i$ of these unacceptable positions, and thus there is $\sum_{i \in N}(k - a_i) = nk - \sum_{i \in N} a_i$ of these positions overall. An unelectable candidate is acceptable for at most $Q-1$ voters and unacceptable for at least $n-Q+1$ voters. Making no candidate electable thus requires $k(n-Q+1)$ unacceptable positions in voters' preferences. Our construction is thus feasible when $k(n-Q+1) \leq nk - \sum_{i \in N} a_i$, or, equivalently, when $Q \geq \frac{\sum_{i \in N} a_i}{k} + 1$. Conversely, if an electorate $\mathbf{v}$ such that $\mathcal{E}(\mathbf{v},Q) = \varnothing$ exists, then it must have candidates in $k(n-Q+1)$ unacceptable positions, which requires that $k(n-Q+1) \leq nk - \sum_{i \in N} a_i$.

Propositions 1 and 2 jointly imply that when $n\frac{k-1}{k} < Q < \frac{\sum_{i \in N} a_i}{k} + 1$, then for any electorate, a stable candidate exists and an electable candidate exists. The question remains as to whether there exists a candidate that is both stable and electable. The following example shows that not necessarily.

Example 2. Consider an election with five voters and three candidates. The voters' preferences $\mathbf{v} = (\succ_1, \succ_2, \succ_3, \succ_4, \succ_5)$ over the candidates are in the following table.

1	1	1	2	3
2	2	2	3	1
3	3	3	1	2

The first column shows the preferences of the first voter with more preferred candidates placed higher, so that her ranking is $1 \succ_1 2 \succ_1 3$, and similarly for the remaining voters. The candidates shown with grey background are unacceptable, so that $(a_i)_{i \in N} = (3,3,3,2,1)$.

[10]Related results in the literature ensure core-nonemptiness (Nakamura, 1979, Theorem 2.3), existence of majority equilibrium (Greenberg, 1979, Corollary 3), acyclicity of simple preference aggregation rules (Austen-Smith and Banks, 2000, Theorem 3.2), or acyclicity of q-quota games (Moulin, 1988, Corollary 1 to Theorem 11.4). We provide a stand-alone proof of Proposition 1 because none of these results immediately applies to our setting (the first one applies to collections of winning coalitions, the second one provides a sufficient condition, the remaining ones require translation of acyclicity and of simple rules or of q-quota games to our setting).

Consider $Q = 4$ and note that $n\frac{k-1}{k} = \frac{10}{3} < Q = \frac{12}{3} < \frac{\sum_{i \in N} a_i}{k} + 1 = \frac{12}{3} + 1$. Propositions 1 and 2 thus imply $\mathcal{S}(\mathbf{v}, Q) \neq \varnothing$ and $\mathcal{E}(\mathbf{v}, Q) \neq \varnothing$. However, no candidate is both stable and electable. To see this, note first that $\mathcal{S}(\mathbf{v}, Q) = \{1\}$. This is because four voters prefer candidate 1 to candidate 2, four voters prefer candidate 2 to candidate 3, but only two voters prefer candidate 3 to candidate 1. Moreover, $\mathcal{E}(\mathbf{v}, Q) = \{2, 3\}$. This is because candidate 1 is the only candidate that is unacceptable for two or more voters.

This example thus shows that ensuring the existence of a stable candidate and ensuring the existence of an electable candidate does not suffice for the existence of a candidate that has both of these properties. Note also that the example is minimal in the sense that candidate 1 becomes both stable and electable once he becomes acceptable for one additional voter. Making candidate 1 acceptable for one additional voter means ensuring either that $a_i = k$ for Q or more voters or that $a_i \geq k - 1$ for all voters. Another way to ensure the existence of stable and electable candidate is to change Q to $Q = n = 5$. In this case candidate 3 has these properties. It is these three conditions that feature in the following proposition along with the conditions from Propositions 1 and 2.

Proposition 3. *$\mathcal{S}(\mathbf{v}, Q) \cap \mathcal{E}(\mathbf{v}, Q) \neq \varnothing$ for all $\mathbf{v} \in \mathbf{V}$ if and only if i) $Q > n\frac{k-1}{k}$, ii) $Q < \frac{\sum_{i \in N} a_i}{k} + 1$, and iii) either $a_i = k$ for Q or more voters, or $Q = n$, or $a_i \geq k - 1$ for all voters.*

Proposition 3 is the main result of this paper. It provides a necessary and sufficient condition for the existence of a candidate that is both stable and electable. The first two conditions ensure that for any electorate $\mathbf{v}$, a stable candidate exists and an electable candidate exists.

How does the third condition ensure that there is a candidate that is both stable and electable? When $a_i = k$ for Q or more voters, each candidate is acceptable for at least Q voters and hence all candidates are electable, including those that are stable.

When $Q = n$, the key observation is that if candidate c is electable but not stable, another candidate c' exists such that all voters prefer c' to c and such that c' is electable. Candidate c' exists because c is not stable and hence there must be a candidate all voters prefer to c. And c' is electable because all voters not only prefer c' to c but also find c acceptable. The observation allows us to find a candidate that is both stable and electable. Starting with electable but unstable candidate c, there is electable candidate c' that all voters prefer to c, and if c' is unstable, then there is electable candidate c'' that all voters prefer to c'. Continuing similarly, this sequence of candidates ends with electable candidate $\tilde{c}$ such that no other candidate is preferred to $\tilde{c}$ by all voters, which makes $\tilde{c}$ stable as well. Note that for each candidate in the sequence, all voters prefer that candidate to all candidates that appear earlier in the sequence, and thus the sequence includes distinct candidates.

When $a_i \geq k-1$ for all voters, each voter has at most one unacceptable candidate. A similar observation as above still applies: if candidate c is electable but not stable, another candidate c' exists such that Q or more voters prefer c' to c and such that c' is electable. Q or more voters prefer c' to c because c is not stable and c' is electable because each of these voters has at most one unacceptable candidate. We can thus construct a similar sequence of candidates as above, one that ends with electable and stable candidate. The added difficulty is making sure that candidates do not repeat in this sequence, which, however, is ruled out by the $Q > n\frac{k-1}{k}$ condition. The condition, as discussed after Proposition 1, implies that there does not exist a sequence of candidates such that for each candidate in the sequence Q or more voters prefer that candidate to the candidate that precedes him, and such that the sequence starts and ends with the same candidate.

When either of the three conditions in Proposition 3 fails, there is an electorate such that no candidate is simultaneously stable and electable. When $Q \leq n\frac{k-1}{k}$, this is because there is an electorate such that no candidate is stable. When $Q \geq \frac{\sum_{i \in N} a_i}{k} + 1$, this is because there is an electorate such that no candidate is electable. When the third condition fails, we can construct an electorate such that no candidate is both stable and electable as in Example 2, which we generalize in Lemma A1. In particular, when the third condition from Proposition 3 fails, $a_i = k$ for $Q-1$ or fewer voters, and $a_i \leq k-2$ for some voter i. These two facts imply that there must be a coalition $C \subseteq N$ of $n-Q+1$ voters such that $a_i \leq k-1$ for all voters $i \in C$, and $a_i \leq k-2$ for some voter $i \in C$. Assigning preferences $1 \succ_i 2 \succ_i \ldots \succ_i k$ to the $Q-1$ voters not in C and preferences $2 \succ_i \ldots \succ_i k \succ_i 1$ to the $n-Q+1$ voters in C, except for the one voter in C with $a_i \leq k-2$ who has $d \succ_i 1 \succ_i 2$, generates an electorate with a unique stable candidate 1 that is not electable.

What if the conditions form Proposition 3 cannot be met and the existence of stable and electable candidates for any electorate is not guaranteed? Can we, for example, say that certain voting rules produce stable and electable candidates for a larger set of electorates? Answering this question is difficult because the set of stable and electable candidates for a given electorate is not monotone in the voting rule, unlike the set of stable candidates, which expands when Q increases, and unlike the set of electable candidates, which expands when Q decreases. Revisiting Example 2, recall that for $Q = 4$ the main message of the example was to show that no candidate is both stable and electable. For both lower or higher values of Q, however, a unique candidate is both stable and electable. Namely, candidate 1 when $Q = 3$ and candidate 3 when $Q = 5$.

The three conditions from Proposition 3 provide some information about the trade-offs involved when the existence of stable and electable candidates cannot be guaranteed. For any given electorate, the first condition provides a non-empty set of stable candidates, the second condition provides a non-empty set of electable candidates, and the third condition, sometimes with the help of the first condition, provides a non-empty intersection of these two sets. Consider the case when the voting rule Q is larger than what the second condition requires (with the other two conditions satisfied). In this case, although the set of electable candidates might be empty, whenever it is not empty, it has a non-empty intersection with the set of stable candidates.[11] Conversely, consider the case when the voting rule Q is smaller than what the first condition requires (with the other two conditions satisfied). In this case, not only is it possible that the set of stable candidates is empty, but also that it is non-empty but has an empty intersection with the set of electable candidates. That is, electorates with a non-empty set of stable candidates and a non-empty set of electable candidates but with an empty intersection of these two sets are in general possible when Q violates the first condition of Proposition 3.[12]

We conclude our discussion of Proposition 3 by highlighting its important corollary. The corollary emerges as a special case of the proposition when there are three candidates and when each voter has two acceptable and one unacceptable candidate, that is, when voters are willing to compromise somewhat. Historical accounts of conclaves as well as anecdotal evidence suggest that both assumptions oftentimes fit the reality of papal elections.

Historically, at a number of conclaves cardinals found themselves split into three groups, with group affiliations generated either by ideology, or by regional alliances, or by the popes who appointed the cardinals. When there are three groups and cardinals' preferences are predominantly driven by group identity, a situation with essentially tree candidates emerges. More recently, in each of the last three conclaves, in 2005, 2013, and 2025, several newspaper reports described the election as having exactly three likely winners. In the latter two conclaves, votes in the first ballot, which is often informative about the serious candidates for papacy, confirmed the tripartite nature of the election. In each of these conclaves, the first ballot saw exactly three candidates getting over 20 votes with these three clearly separated from the rest of the field.[13]

[11] Careful reading of the proof of Proposition 3 shows that if the first and the third condition of the proposition hold, then for any electorate with a non-empty set of electable candidates, the set of stable and electable candidates is non-empty.

[12] We can construct $\mathbf{v}$ such that $\mathcal{S}(\mathbf{v}, Q) \neq \varnothing$ and $\mathcal{E}(\mathbf{v}, Q) \neq \varnothing$ but $\mathcal{S}(\mathbf{v}, Q) \cap \mathcal{E}(\mathbf{v}, Q) = \varnothing$ when $Q \leq n\frac{k-2}{k-1}$, $Q < \frac{\sum_{i \in N} a_i}{k} + 1$ and $a_i = k-1$ for all voters (i.e., the first condition from Proposition 3 fails but the other two conditions hold) under an auxiliary condition $Q > \frac{n+1}{2}$. First, we make candidates 1 through $k-1$ unstable by allocating each of the preference fragments $1 \succ_i 2$, $2 \succ_i 3$, $\ldots$, $k-2 \succ_i k-1$ and $k-1 \succ_i 1$ to Q voters. The construction is discussed after Proposition 1, formally described in its proof, and requires $Q \leq n\frac{k-2}{k-1}$. Second, the remaining candidate k is assigned to be the least preferred for $n - Q + 1$ voters and the most preferred for all remaining voters. This assignment makes candidate k stable, because for any other candidate $n - Q + 1 < Q$ voters prefer that candidate to k, but unelectable, because he is acceptable for $Q - 1$ voters. The unique stable candidate k is unelectable and all the remaining candidates are unstable but at least one of them has to be electable because $Q < \frac{\sum_{i \in N} a_i}{k} + 1$.

[13] For descriptions of conclaves with cardinals split into three groups see Baumgartner (2003, p. 128, 133, 183, 188). The 2005 conclave: The New York Times: Holy Rollers and Papal Perfectas, April 18, 2005 and The Atlantic Monthly: The Year of Two Popes, Jan/Feb 2006. The 2013 conclave: The Vancouver Sun: Canada's Marc Ouellet Came 'Very Close' to Becoming Pope, March 16, 2013 and The Christian Science Monitor: Who's on the Short List to Be the New Pope, March 12, 2013. The 2025

Compromise is a thread that runs through the conclaves. That conclaves last at least few days shows that cardinals typically have heterogeneous preferences and do not compromise readily. That conclaves end shows that cardinals compromise eventually. The key feature of the conclave, locking up cardinals and not allowing them to leave until they choose a pope, is meant to enforce compromise. But reaching compromise might take time; 17th and 18th centuries have witnessed at least four conclaves that lasted more than 100 days. In at least one instance, cardinals claimed that 'they were willing to die in the conclave before voting for a candidate for the other side'.[14]

Corollary 1. *Suppose $k = 3$ and $a_i = 2$ for each voter $i \in N$. A candidate that is both stable and electable exists for any electorate $\mathbf{v} \in \mathbf{V}$ if and only if $\frac{2}{3}n < Q < \frac{2}{3}n + 1$.*

When there are three candidates and voters are willing to compromise somewhat, then the voting rule Q that guarantees the existence of a stable and electable candidate is the integer strictly between $\frac{2}{3}n$ and $\frac{2}{3}n+1$. When n is not divisible by three, the unique such integer is $\frac{2}{3}n$ rounded up. That is, it is the 2/3 supermajority Catholic Church has been using for almost a millennium. The corollary confirms that the 2/3 supermajority is the right compromise between electability and stability, because it ensures both. And it is a unique rule that ensures both.[15]

4 Concluding remarks

It is obvious that an election without an electable candidate will remain in eternal voting. And we would argue that, in an election where just one candidate is electable, the outcome will be to elect such candidate. But the results of this paper are silent regarding how, when, and who will be elected when several candidates (stable or not) are electable. Attempting to make such predictions requires behavioural and strategic considerations that are outside the scope of the present paper. An explicit model of the election as a dynamic non-cooperative game is necessary. We do that in a companion paper. In Ponsatí and Zápal (2025) we explicitly model repeated voting as a non-cooperative game and examine the (equilibrium) process whereby voters, after supporting their top candidate at the onset, maintain such support or switch to vote other acceptable candidates as the

conclave: New York Times: How a Quiet American Cardinal Became Pope, May 11, 2025. The first ballot in the 2013 conclave reported by O'Connell (2021): Scola 30, Bergoglio 26, Ouellet 22, O'Malley 10, all other cardinals fewer votes. The first ballot in the 2025 conclave reported by O'Connell and Piqué (2025): Erdö, Prevost and Parolin above 20 votes, Aveline between 10 and 20 votes, all other cardinals fewer votes.

[14] Baumgartner (2003, ch. 8) describes conclaves in the 17th and 18th century. Conclaves that lasted more than 100 days took place in 1669, 1691, 1730 and 1740. The citation about the unwillingness to compromise comes from the 1549 conclave, which lasted more than 70 days (Baumgartner, 2003, p. 109).

[15] When n is divisible by three, no integer satisfies $\frac{2}{3}n < Q < \frac{2}{3}n + 1$. We discuss above that $Q = \frac{2}{3}n$ ensures electability and $Q = \frac{2}{3}n + 1$ ensures stability.

rounds of voting ensue. The tools and results of the present paper are useful building blocks for that analysis.

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A Proofs

Proof of Lemma 1

Fix $\mathbf{v} \in \mathbf{V}$, $Q, Q' \in \mathbb{N}$ such that $Q < Q'$, and $c \in K$.

To prove that $\mathcal{E}(\mathbf{v}, Q') \subseteq \mathcal{E}(\mathbf{v}, Q)$, it suffices to prove that if $c \in \mathcal{E}(\mathbf{v}, Q')$, then $c \in \mathcal{E}(\mathbf{v}, Q)$. If $c \in \mathcal{E}(\mathbf{v}, Q')$, then $|\{i \in N : c \succ_i d\}| \geq Q'$, and hence $|\{i \in N : c \succ_i d\}| \geq Q' > Q$, which implies $c \in \mathcal{E}(\mathbf{v}, Q)$.

To prove that $\mathcal{S}(\mathbf{v}, Q) \subseteq \mathcal{S}(\mathbf{v}, Q')$, it suffices to prove that if $c \in \mathcal{S}(\mathbf{v}, Q)$, then $c \in \mathcal{S}(\mathbf{v}, Q')$. If $c \in \mathcal{S}(\mathbf{v}, Q)$, then $|\{i \in N : c' \succ_i c\}| < Q \ \forall c' \in K \setminus \{c\}$, and hence $|\{i \in N : c' \succ_i c\}| < Q < Q' \ \forall c' \in K \setminus \{c\}$, which implies $c \in \mathcal{S}(\mathbf{v}, Q')$. □

Proof of Proposition 1

We prove that $\mathcal{S}(\mathbf{v}, Q) = \varnothing$ for some $\mathbf{v} \in \mathbf{V}$ if and only if $Q \leq n\frac{k-1}{k}$.

Only if: Fix $\mathbf{v} \in \mathbf{V}$ such that $\mathcal{S}(\mathbf{v}, Q) = \varnothing$. We show that $Q \leq n\frac{k-1}{k}$. Because $\mathcal{S}(\mathbf{v}, Q) = \varnothing$, there exists a sequence of $m \leq k$ distinct candidates $(c_j)_{j=1}^m$ such that, using $c_{m+1} = c_1$, we have $|\{i \in N : c_{j+1} \succ_i c_j\}| \geq Q \ \forall j \in \{1, \ldots, m\}$. Because $n\frac{m-1}{m} \leq n\frac{k-1}{k}$ when $m \leq k$, it suffices to show that $Q \leq n\frac{m-1}{m}$, which we do by proving two claims. First, we claim that $\sum_{j=1}^m |\{i \in N : c_{j+1} \succ_i c_j\}| \geq mQ$, which follows from $|\{i \in N : c_{j+1} \succ_i c_j\}| \geq Q \ \forall j \in \{1, \ldots, m\}$. Second, we claim that $\sum_{j=1}^m |\{i \in N : c_{j+1} \succ_i c_j\}| \leq n(m-1)$. To see this, let $\mathbb{I}(s) = 1$ if s is true and $\mathbb{I}(s) = 0$ if s is false, and note that $\sum_{j=1}^m \mathbb{I}(c_{j+1} \succ_i c_j) \leq m - 1 \ \forall i \in N$ because $\sum_{j=1}^m \mathbb{I}(c_{j+1} \succ_i c_j) = m$ for some $i \in N$ would imply by transitivity of $\succ_i$ that $c_{m+1} \succ_i c_1 = c_{m+1}$. Thus, $\sum_{j=1}^m |\{i \in N : c_{j+1} \succ_i c_j\}| = \sum_{j=1}^m \sum_{i \in N} \mathbb{I}(c_{j+1} \succ_i c_j) = \sum_{i \in N} \sum_{j=1}^m \mathbb{I}(c_{j+1} \succ_i c_j) \leq \sum_{i \in N} (m-1) = n(m-1)$. The two claims imply $mQ \leq n(m-1)$, or, equivalently, $Q \leq n\frac{m-1}{m}$.

If: Suppose $Q \leq n\frac{k-1}{k}$. We show that $\mathbf{v} \in \mathbf{V}$ such that $\mathcal{S}(\mathbf{v}, Q) = \varnothing$ exists. Denote $L_c = \{(c-1)(n-Q)+1, \ldots, (c-1)(n-Q)+n-Q\}$ $\forall c \in K$. Define $c' \in K$ implicitly by $n \in L_{c'}$. Note that c' exists because $\cup_{c\in K} L_c = \{1, \ldots, k(n-Q)\}$ and because $1 \leq n \leq k(n-Q)$, where the second inequality follows from $Q \leq n\frac{k-1}{k}$. Moreover, c' is unique because for any $j, j' \in K$ such that $j \neq j'$, $L_j \cap L_{j'} = \varnothing$. Allocate the voters into sets D_c for all $c \in K$ such that

$$\begin{aligned} &D_c = \{i \in N : i \in L_c\} && \text{if } c < c' \\ &D_c = \{i \in N : i \in L_c \ \wedge \ i \leq n\} && \text{if } c = c' \\ &D_c = \varnothing && \text{if } c > c'. \end{aligned} \tag{A1}$$

Note that, by construction, $\cup_{c\in K} D_c = N$, $D_j \cap D_{j'} = \varnothing$ for any $j, j' \in K$ such that $j \neq j'$, and $|N \setminus D_c| \geq Q$ $\forall c \in K$, which follows from $|L_c| \leq n - Q$ $\forall c \in K$.

There exists $\mathbf{v} = (\succ_i)_{i\in N}$ such that

$$\begin{array}{llllllllll} \forall i \in D_1, & k & \succ_i & k-1 & \succ_i & \ldots & \ldots & \ldots \succ_i & 2 & \succ_i \ 1 \\ \forall i \in D_2, & 1 & \succ_i & k & \succ_i & \ldots & \ldots & \ldots \succ_i & 3 & \succ_i \ 2 \\ \forall i \in D_3, & 2 & \succ_i & 1 & \succ_i & \ldots & \ldots & \ldots \succ_i & 4 & \succ_i \ 3 \\ \quad\vdots & & & & & & & & & \\ \forall i \in D_{c+1}, & c & \succ_i & c-1 & \succ_i & \ldots & \succ_i 1 \succ_i k \succ_i & \ldots \succ_i & c+2 & \succ_i \ c+1 \\ \quad\vdots & & & & & & & & & \\ \forall i \in D_k, & k-1 & \succ_i & k-2 & \succ_i & \ldots & \ldots & \ldots \succ_i & 1 & \succ_i \ k. \end{array} \tag{A2}$$

Given this $\mathbf{v}$, $|\{i \in N : 1 \succ_i k\}| = |N \setminus D_1| \geq Q$, and hence $k \notin \mathcal{S}(\mathbf{v}, Q)$, as well as, $\forall c \in K \setminus \{k\}$, $|\{i \in N : c+1 \succ_i c\}| = |N \setminus D_{c+1}| \geq Q$, and hence $c \notin \mathcal{S}(\mathbf{v}, Q)$. Thus $\mathcal{S}(\mathbf{v}, Q) = \varnothing$. □

Proof of Proposition 2

We prove that $\mathcal{E}(\mathbf{v}, Q) = \varnothing$ for some $\mathbf{v} \in \mathbf{V}$ if and only if $Q \geq \frac{\sum_{i\in N} a_i}{k} + 1$.

Only if: Fix $\mathbf{v} \in \mathbf{V}$ such that $\mathcal{E}(\mathbf{v}, Q) = \varnothing$. We show that $Q \geq \frac{\sum_{i\in N} a_i}{k} + 1$ by proving two claims. First, we claim that $\sum_{c\in K} |\{i \in N : c \succ_i d\}| = \sum_{i\in N} a_i$. To see this, let $\mathbb{I}(s) = 1$ if s is true and $\mathbb{I}(s) = 0$ if s is false. Then $\sum_{c\in K} |\{i \in N : c \succ_i d\}| = \sum_{c\in K} \sum_{i\in N} \mathbb{I}(c \succ_i d) = \sum_{i\in N} \sum_{c\in K} \mathbb{I}(c \succ_i d) = \sum_{i\in N} a_i$. Second, we claim that $\sum_{c\in K} |\{i \in N : c \succ_i d\}| \leq k(Q-1)$. To see this, note that $\mathcal{E}(\mathbf{v}, Q) = \varnothing$ implies $|\{i \in N : c \succ_i d\}| \leq Q - 1$ $\forall c \in K$. The two claims imply $\sum_{i\in N} a_i \leq k(Q-1)$, or, equivalently, $Q \geq \frac{\sum_{i\in N} a_i}{k} + 1$.

If: Suppose $Q \geq \frac{\sum_{i \in N} a_i}{k} + 1$. We show that $\mathbf{v} \in \mathbf{V}$ such that $\mathcal{E}(\mathbf{v}, Q) = \varnothing$ exists. Construct sequence of candidates

$$C = (c_j)_{j=1}^{k(n-Q+1)} = (\underbrace{1, \ldots, k, 1, \ldots, k, \ \ldots \ldots \ldots, 1, \ldots, k}_{(1,\ldots,k) \text{ repeated } n-Q+1 \text{ times}}). \tag{A3}$$

Denote $M_i = \{(\sum_{j=1}^{i-1} k - a_j) + 1, \ldots, (\sum_{j=1}^{i-1} k - a_j) + k - a_i\}$ $\forall i \in N$. Define $i' \in N$ implicitly by $k(n - Q + 1) \in M_{i'}$. Note that i' exists because $\cup_{i \in N} M_i = \{1, \ldots, nk - \sum_{i \in N} a_i\}$ and because $1 \leq k(n - Q + 1) \leq nk - \sum_{i \in N} a_i$, where the second inequality follows from $Q \geq \frac{\sum_{i \in N} a_i}{k} + 1$. Moreover, i' is unique because for any $j, j' \in N$ such that $j \neq j'$, $M_j \cap M_{j'} = \varnothing$. Allocate the elements (i.e., the candidates) of $C = (c_j)_{j=1}^{k(n-Q+1)}$ into sets R_i for all $i \in N$ such that

$$\begin{aligned}
&R_i = \{c_j \in K : j \in M_i\} && \text{if } i < i' \\
&R_i = \{c_j \in K : j \in M_i \ \wedge \ j \leq k(n - Q + 1)\} && \text{if } i = i' \\
&R_i = \varnothing && \text{if } i > i'.
\end{aligned} \tag{A4}$$

Note that, $\forall i \in N$, $|R_i| \leq k - a_i$ because $|M_i| = k - a_i$. Moreover, $\forall c \in K$, $|\{i \in N : c \in R_i\}| = n - Q + 1$, which is because, $\forall i \in N$, M_i is a set of $k - a_i \leq k$ consecutive integers, because any k or fever consecutive elements of C are all distinct, and because C includes each candidate exactly $n - Q + 1$ times.

Therefore, $\mathbf{v} = (\succ_i)_{i \in N}$ such that $R_i \subseteq \{c \in K : d \succ_i c\}$ $\forall i \in N$ exists. Given this $\mathbf{v}$, $\forall i \in N$ and $\forall c \in K$, if $c \in R_i$, then $d \succ_i c$. Thus, $\forall c \in K$, $|\{i \in N : d \succ_i c\}| \geq |\{i \in N : c \in R_i\}|$, and hence $|\{i \in N : c \succ_i d\}| = n - |\{i \in N : d \succ_i c\}| \leq n - |\{i \in N : c \in R_i\}| = n - (n - Q + 1) = Q - 1$. Thus $\mathcal{E}(\mathbf{v}, Q) = \varnothing$. □

Proof of Proposition 3

If: Fix $\mathbf{v} \in \mathbf{V}$. We start by establishing two claims. First, assuming either $Q = n$ or $a_i \geq k - 1$ $\forall i \in N$, we claim that if $c \in \mathcal{E}(\mathbf{v}, Q)$ and $|\{i \in N | c' \succ_i c\}| \geq Q$ for some $c, c' \in K$, then $c' \in \mathcal{E}(\mathbf{v}, Q)$. To see this, fix $c, c' \in K$ such that $|\{i \in N : c \succ_i d\} \geq Q$ and $|\{i \in N : c' \succ_i c\}| \geq Q$. If $Q = n$, we have, $\forall i \in N$, $c \succ_i d$ and $c' \succ_i c$ and hence $c' \succ_i d$. Thus $c' \in \mathcal{E}(\mathbf{v}, Q)$. If $a_i \geq k - 1$ $\forall i \in N$, each voter has at most one unacceptable candidate and thus if $c' \succ_i c$ for some $i \in N$, then $c' \succ_i d$. Therefore, $|\{i \in N : c' \succ_i c\}| \geq Q$ implies $c' \in \mathcal{E}(\mathbf{v}, Q)$.

Second, assuming $Q > n\frac{k-1}{k}$, we claim that for any sequence of $m \leq k$ distinct candidates $(c_j)_{j=1}^m$, using $c_{m+1} = c_1$, $|\{i \in N | c_{j+1} \succ_i c_j\}| \geq Q$ $\forall j \in \{1, \ldots, m\}$ is not possible. Suppose, towards a contradiction, that such a sequence exists. Then $mQ \leq \sum_{j=1}^m |\{i \in N | c_{j+1} \succ_i c_j\}| \leq n(m - 1)$. The first inequality follows from $|\{i \in N : c_{j+1} \succ_i c_j\}| \geq Q$ $\forall j \in \{1, \ldots, m\}$. To see the second inequality, let $\mathbb{I}(s) = 1$ if s is true and

$\mathbb{I}(s) = 0$ if s is false. Then, $\forall i \in N$, $\sum_{j=1}^{m} \mathbb{I}(c_{j+1} \succ_i c_j) \le m-1$ because $\succ_i$ is transitive. Hence $\sum_{j=1}^{m} |\{i \in N | c_{j+1} \succ_i c_j\}| = \sum_{j=1}^{m} \sum_{i \in N} \mathbb{I}(c_{j+1} \succ_i c_j) = \sum_{i \in N} \sum_{j=1}^{m} \mathbb{I}(c_{j+1} \succ_i c_j) \le n(m-1)$. The two inequalities jointly imply $Q \le n\frac{m-1}{m}$. Because $m \le k$, $\frac{m-1}{m} \le \frac{k-1}{k}$ and thus $Q \le n\frac{m-1}{m} \le n\frac{k-1}{k}$, which is a contradiction to $Q > n\frac{k-1}{k}$.

Suppose now that $n\frac{k-1}{k} < Q < \frac{\sum_{i \in N} a_i}{k} + 1$. Then $\mathcal{S}(\mathbf{v}, Q) \neq \varnothing$ by Proposition 1 and $\mathcal{E}(\mathbf{v}, Q) \neq \varnothing$ by Proposition 2. If $a_i = k$ for Q or more voters, then $\mathcal{E}(\mathbf{v}, Q) = K$, and hence $\mathcal{E}(\mathbf{v}, Q) \cap \mathcal{S}(\mathbf{v}, Q) \neq \varnothing$, as desired. To cover the remaining two cases, suppose either $Q = n$ or $a_i \ge k-1$ $\forall i \in N$, and, towards a contradiction, that $\mathcal{E}(\mathbf{v}, Q) \cap \mathcal{S}(\mathbf{v}, Q) = \varnothing$. $\mathcal{E}(\mathbf{v}, Q) \neq \varnothing$, $\mathcal{S}(\mathbf{v}, Q) \neq \varnothing$ and the contradiction assumption imply that $c_1 \in K$ exists such that $c_1 \notin \mathcal{S}(\mathbf{v}, Q)$ and $c_1 \in \mathcal{E}(\mathbf{v}, Q)$.

Because $c_1 \notin \mathcal{S}(\mathbf{v}, Q)$, $c_2 \in K$ exists such that $|\{i \in N | c_2 \succ_i c_1\}| \ge Q$, which, by the first claim and $c_1 \in \mathcal{E}(\mathbf{v}, Q)$, implies $c_2 \in \mathcal{E}(\mathbf{v}, Q)$, and hence, by the contradiction assumption, $c_2 \notin \mathcal{S}(\mathbf{v}, Q)$. By construction, $c_2 \neq c_1$. Hence c_1 and c_2 are distinct candidates, both belong to $\mathcal{E}(\mathbf{v}, Q)$ and neither belongs to $\mathcal{S}(\mathbf{v}, Q)$. If $k = 2$, this is a contradiction to $\mathcal{S}(\mathbf{v}, Q) \neq \varnothing$. Hence $k \ge 3$.

Because $c_2 \notin \mathcal{S}(\mathbf{v}, Q)$, $c_3 \in K$ exists such that $|\{i \in N | c_3 \succ_i c_2\}| \ge Q$, which, by the first claim and $c_2 \in \mathcal{E}(\mathbf{v}, Q)$, implies $c_3 \in \mathcal{E}(\mathbf{v}, Q)$, and hence, by the contradiction assumption, $c_3 \notin \mathcal{S}(\mathbf{v}, Q)$. By construction, $c_3 \neq c_2$. By the second claim, $c_3 \neq c_1$ (if $c_3 = c_1$, then $(c_j)_{j=1}^{2}$, using $c_3 = c_1$, would constitute the sequence the second claim shows cannot exist). Hence c_1, c_2 and c_3 are distinct candidates, all belong to $\mathcal{E}(\mathbf{v}, Q)$ and neither belongs to $\mathcal{S}(\mathbf{v}, Q)$. If $k = 3$, this is a contradiction to $\mathcal{S}(\mathbf{v}, Q) \neq \varnothing$. Hence $k \ge 4$.

Continuing analogously, $k-1$ distinct candidates c_1, c_2, $\dots$, c_{k-1} exist, all belong to $\mathcal{E}(\mathbf{v}, Q)$, neither belongs to $\mathcal{S}(\mathbf{v}, Q)$, and $|\{i \in N : c_{j+1} \succ_i c_j\}| \ge Q$ $\forall j \in \{1, \dots, k-2\}$. Because $c_{k-1} \notin \mathcal{S}(\mathbf{v}, Q)$, $c_k \in K$ exists such that $|\{i \in N | c_k \succ_i c_{k-1}\}| \ge Q$, which, by the first claim and $c_{k-1} \in \mathcal{E}(\mathbf{v}, Q)$, implies $c_k \in \mathcal{E}(\mathbf{v}, Q)$, and hence, by the contradiction assumption, $c_k \notin \mathcal{S}(\mathbf{v}, Q)$. By construction, $c_k \neq c_{k-1}$. By the second claim, $c_k \neq c_1$ (if $c_k = c_1$, then $(c_j)_{j=1}^{k-1}$, using $c_k = c_1$, would constitute the sequence the second claim shows cannot exist), and similarly, $c_k \neq c_j$ $\forall j \in \{2, \dots, k-2\}$. Hence c_1, c_2, $\dots$, c_k are distinct candidates and neither belongs to $\mathcal{S}(\mathbf{v}, Q)$. Thus $\mathcal{S}(\mathbf{v}, Q) = \varnothing$, which is a contradiction.

Only if: If $Q \le n\frac{k-1}{k}$, there exists $\mathbf{v} \in \mathbf{V}$ such that $\mathcal{S}(\mathbf{v}, Q) = \varnothing$ by Proposition 1. If $Q \ge \frac{\sum_{i \in N} a_i}{k} + 1$, there exists $\mathbf{v} \in \mathbf{V}$ such that $\mathcal{E}(\mathbf{v}, Q) = \varnothing$ by Proposition 2. Therefore, what remains to prove is that there exists $\mathbf{v} \in \mathbf{V}$ such that $\mathcal{E}(\mathbf{v}, Q) \cap \mathcal{S}(\mathbf{v}, Q) = \varnothing$ when $n\frac{k-1}{k} < Q < \frac{\sum_{i \in N} a_i}{k} + 1$, $Q \le n-1$, $a_i = k$ for $Q-1$ or fewer voters, and $a_i \le k-2$ for some voter $i \in N$. We now argue that Lemma A1, which we state and prove below, applies under these conditions. This concludes the proof because the lemma constructs $\mathbf{v} \in \mathbf{V}$ such that $\mathcal{E}(\mathbf{v}, Q) \cap \mathcal{S}(\mathbf{v}, Q) = \varnothing$.

We make a series of claims to establish that Lemma A1 applies when $n\frac{k-1}{k} < Q <$

$\frac{\sum_{i \in N} a_i}{k} + 1$, $Q \leq n-1$, $a_i = k$ for $Q-1$ or fewer voters, and $a_i \leq k-2$ for some voter $i \in N$. First, note that $\frac{n+1}{2} \leq Q$ and $Q \leq n-1$ when $n = 2$ imply $\frac{3}{2} \leq Q \leq 1$, which is not possible, and hence $n \geq 3$. Second, $\frac{n+1}{2} \leq n\frac{k-1}{k}$ is equivalent to $1 \leq n\frac{k-2}{k}$, which holds when $n \geq 3$ and $k \geq 3$. That is, because $Q > n\frac{k-1}{k}$, we have either $k = 2$ or $Q > \frac{n+1}{2}$. Third, because $a_i = k$ for $Q-1$ or fewer voters and $a_i \in \{0, \ldots, k\}$ $\forall i \in N$, $a_i \leq k-1$ for $n-Q+1$ or more voters. Moreover, $a_i \leq k-2$ for some $i \in N$. Therefore, there exists $C \subseteq N$ such that $|C| = n-Q+1$, $a_i \leq k-1$ $\forall i \in C$, and $a_i \leq k-2$ for some $i \in C$. □

Lemma A1. *Suppose $Q \leq n-1$ and either $k = 2$ or $Q > \frac{n+1}{2}$. Suppose there exists $C \subseteq N$ such that $|C| = n-Q+1$, $a_i \leq k-1$ $\forall i \in C$, and $a_i \leq k-2$ for some $i \in C$. There exists $\mathbf{v} \in \mathbf{V}$ such that $\mathcal{E}(\mathbf{v}, Q) \cap \mathcal{S}(\mathbf{v}, Q) = \varnothing$.*

Proof. Suppose $Q \leq n-1$ and either $k = 2$ or $Q > \frac{n+1}{2}$. Fix $C \subseteq N$ and $i' \in C$ such that $|C| = n-Q+1$, $a_i \leq k-1$ $\forall i \in C$, and $a_{i'} \leq k-2$. We show that $\mathbf{v} \in \mathbf{V}$ exists such that $\mathcal{S}(\mathbf{v}, Q) = \{1\}$ and $1 \notin \mathcal{E}(\mathbf{v}, Q)$.

Consider $\mathbf{v} \in \mathbf{V}$ such that $1 \succ_i 2 \succ_i \ldots \succ_i k$ $\forall i \in N \setminus C$, $2 \succ_i 3 \succ_i \ldots \succ_i k \succ_i 1$ $\forall i \in C \setminus \{i'\}$, and $d \succ_{i'} 1 \succ_{i'} 2$, which exists. The number of voters in the three sets $N \setminus C$, $C \setminus \{i'\}$ and $\{i'\}$ is, respectively, $Q-1$, $n-Q$ and 1. Moreover, $\forall i \in C \setminus \{i'\}$, $a_i \leq k-1$ and thus $d \succ_i 1$.

Given this $\mathbf{v}$, $|\{i \in N : 1 \succ_i d\}| \leq |N \setminus C| = Q-1 < Q$, and hence $1 \notin \mathcal{E}(\mathbf{v}, Q)$. Moreover, $|\{i \in N : 1 \succ_i 2\}| = |N \setminus C| + |\{i'\}| = Q$ and $|\{i \in N : 2 \succ_i 1\}| = n-Q < Q$, where the inequality follows from $Q \geq \frac{n+1}{2}$. If $k = 2$, we thus have $\mathcal{S}(\mathbf{v}, Q) = \{1\}$, as desired. If $Q > \frac{n+1}{2}$, $\mathcal{S}(\mathbf{v}, Q) = \{1\}$ follows because we additionally have, $\forall c \in K \setminus \{1, k\}$, $|\{i \in N : c \succ_i c+1\}| \geq |N \setminus C| + |C \setminus \{i'\}| = n-1 \geq Q$, as well as, $\forall c \in K \setminus \{1, 2\}$, $|\{i \in N : c \succ_i 1\}| \leq |C| = n-Q+1 < Q$, where the second inequality follows from $Q > \frac{n+1}{2}$. □

Abstrakt

Konečná skupina voličů vybírá papeže z konečné skupiny kandidátů. Počet kol volby není omezen a voliči hlasují do okamžiku, kdy jeden z kandidátů získá Q hlasů. Kandidát je volitelný, pakliže jej dostatek voličů preferuje nekonečnému opakování kol volby. Kandidát je stabilní, pakliže neexistuje jiný kandidát, kterého by dostatek voličů preferoval. Dokazujeme nutnou a postačující podmínku pro existenci volitelného a stabilního kandidáta. Pakliže jsou kandidáti tři a voliči jsou částečně otevřeni kompromisu, pak tato podmínka vyžaduje volbu pomocí dvoutřetinové většiny hlasů, což je způsob, který katolická církev používá k volbě papeže již po téměř tisíc let.

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